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Computer-assisted existence proofs for one-dimensional Schrödinger-Poisson systems

Jonathan Wunderlich

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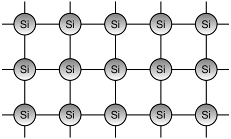
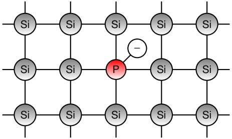
Department of Mathematics, Institute for Analysis

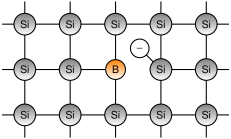
Outline

- Schrödinger-Poisson systems
- Theory of computer-assisted proofs
  - Existence and enclosure theorem
- Application to the one-dimensional Schrödinger-Poisson system
  - Approximate solution $\tilde{u}$
  - Defect bound δ
  - Norm bound for L^{-1}
  - Non-decreasing function g
  - Results

The three-dimensional time-dependent Schrödinger-Poisson system







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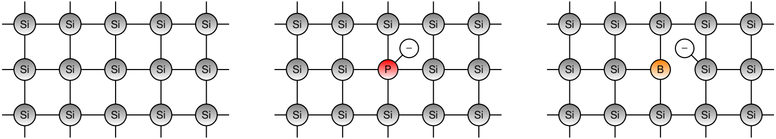

$$i\hbar\partial_t\psi = -\frac{\hbar^2}{2m}\Delta\psi + W\psi \qquad \text{on } [0,\infty)\times\mathbb{R}^3$$

- wavefunction ψ, mass m, Planck's constant $\hbar$

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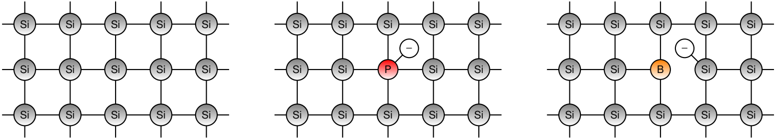

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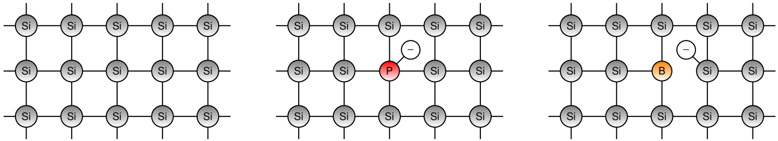

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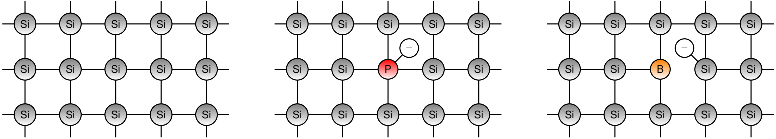

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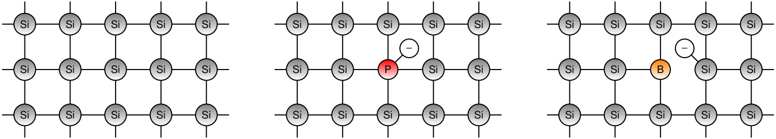

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The one-dimensional Schrödinger-Poisson system

Goal:
Non-trivial solutions of the one-dimensional stationary Schrödinger-Poisson system

$$\left.\begin{aligned} -u'' + V_0 u + W_u u &= u^3 \\ -W_u'' + cW_u &= u^2 \end{aligned}\right\} \text{ on } \mathbb{R}$$
$$\lim_{x\to\pm\infty} \Phi_u = 0$$

- parameter $c > 0$
- constant external potential $V_0 > 0$

The one-dimensional Schrödinger-Poisson system

Using the Green's function Γ for $-W'' + cW$:

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- $\langle u, v \rangle_{\mathrm{H}^1} := \langle u', v' \rangle_{\mathrm{L}^2} + \sigma \langle u, v \rangle_{\mathrm{L}^2} \ (u, v \in \mathrm{H}^1(\mathbb{R}))$, $\sigma > 0$ fixed

Setting

Define $F\colon \mathrm{H}^1_S(\mathbb{R}) \to \mathrm{H}^{-1}_S(\mathbb{R})$ by

$$(Fu)[v] := \int_{\mathbb{R}} \left[u'v' + \left(V_0 + \int_{\mathbb{R}} \Gamma(\cdot, t) u(t)^2 \, dt \right) uv - u^3 v \right] dx$$

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Linearization of F at $\tilde{u}$:

$$L\colon \mathrm{H}^1_S(\mathbb{R}) \to \mathrm{H}^{-1}_S(\mathbb{R}), \; L = F'\tilde{u}$$

Assumptions

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Need constants $\delta \geq 0$, $K \geq 0$ and a non-decreasing function $g\colon [0,\infty) \to [0,\infty)$ satisfying:

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- $\left\|F'(\tilde{u}+u) - F'\tilde{u}\right\|_{\mathcal{B}} \leq g(\|u\|_{\mathrm{H}^1}) \quad \left(u \in \mathrm{H}^1_S(\mathbb{R})\right)$ (A3)
- $g(t) \to 0 \ (t \to 0^+)$ (A4)

Existence and enclosure theorem

Theorem (Existence and enclosure theorem, see [4])

Let $\tilde{u} \in \mathrm{H}^1_S(\mathbb{R})$ *be an approximate solution of* $Fu = 0$*, i.e. of (SPS). Moreover let* $\tilde{u}, \delta, K$ *and* $g \colon [0, \infty) \to [0, \infty)$ *satisfy the assumptions (A1) - (A4).*

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Suppose some $\alpha \geq 0$ *exists, such that*

$$\delta \leq \frac{\alpha}{K} - G(\alpha) \quad \text{and}$$
$$K \cdot g(\alpha) < 1,$$

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Then there exists an exact solution $u^* \in \mathrm{H}^1_S(\mathbb{R})$ *of* $Fu = 0$*, satisfying the enclosure*

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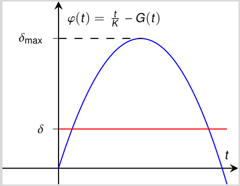

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(A0) Approximate solution $\tilde{u}$

Look for approximations in $V^S_{R,M} = \operatorname{span}\left\{\varphi^S_{R,k} : 1 \leq k \leq M\right\} \subset \mathrm{H}^1_S(\mathbb{R})$ with

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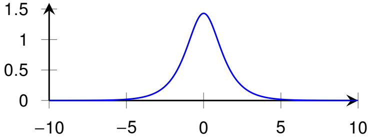
Start Newton method with
$p = 0$ and $u = \frac{\sqrt{2V_0}}{\cosh(\sqrt{V_0}\cdot)}$



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Thus for $\varphi \in H^1_S(\mathbb{R})$

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Verified computation:

- $|\tilde{u}'(R)| + |\tilde{u}'(-R)|$
- $\left\|-\tilde{u}'' + \left(V_0 + \int_{-R}^{R}\Gamma(\cdot,t)\tilde{u}(t)^2\,dt\right)\tilde{u} - \tilde{u}^3\right\|_{L^2(-R,R)}$

(A2) Norm bound for L^{-1}

Norm bound $K \geq 0$ for L^{-1}:

$$\|u\|_{\mathrm{H}^1} \leq K \|Lu\|_{\mathrm{H}^{-1}} \qquad \left(u \in \mathrm{H}^1_S(\mathbb{R})\right) \tag{A2}$$

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Spectral decomposition of $\Phi^{-1} L$ yields:

$$\text{(A2) holds} \quad \Leftrightarrow \quad \gamma := \min\{|\lambda|\colon \lambda \in \sigma\left(\Phi^{-1} L\right)\} > 0$$

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Compute verified lower bounds:

- $\gamma_{\mathrm{ev}} > 0$ for $\min\{|\lambda|\colon \lambda$ isolated eigenvalue of $\Phi^{-1}L\}$
- $\gamma_{\mathrm{ess}} > 0$ for $\min\{|\lambda|\colon \lambda \in \sigma_{\mathrm{ess}}(\Phi^{-1}L)\}$

(A2) Norm bound for L^{-1}

Eigenvalue bound γ_{ev}:

$$\Phi^{-1} L u = \lambda u \underset{\kappa := \frac{1}{1-\lambda}}{\Leftrightarrow} \langle u, \varphi \rangle_{\mathrm{H}^1} = \kappa(\Phi u - L u)[\varphi] \; (\varphi \in \mathrm{H}^1(\mathbb{R}))$$

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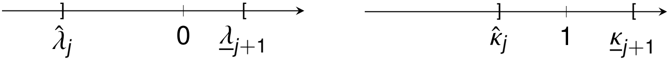

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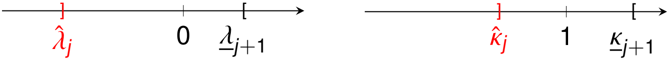

Upper eigenvalue bounds: Rayleigh-Ritz method

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Lower eigenvalue bounds: Lehmann-Goerisch and homotopy method

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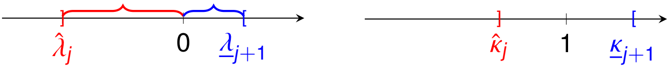


Upper eigenvalue bounds: Rayleigh-Ritz method

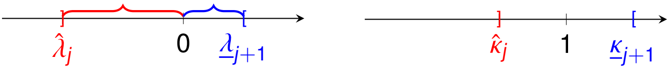
Lower eigenvalue bounds: Lehmann-Goerisch and homotopy method

Set $\gamma_{\text{ev}} = \min\{|\hat{\lambda}_j|, \underline{\lambda}_{j+1}\} = \min\{|1 - \frac{1}{\hat{\kappa}_j}|, 1 - \frac{1}{\underline{\kappa}_{j+1}}\}$

(A2) Norm bound for L^{-1}

Eigenvalue bound γ_{ev}:

$$\Phi^{-1} L u = \lambda u \underset{\kappa := \frac{1}{1-\lambda}}{\Leftrightarrow} \langle u, \varphi \rangle_{H^1} = \kappa(\Phi u - L u)[\varphi] \ (\varphi \in H^1(\mathbb{R}))$$



Upper eigenvalue bounds: Rayleigh-Ritz method

Lower eigenvalue bounds: Lehmann-Goerisch and homotopy method

Set $\gamma_{\text{ev}} = \min\{|\hat{\lambda}_j|, \underline{\lambda}_{j+1}\} = \min\{|1 - \frac{1}{\hat{\kappa}_j}|, 1 - \frac{1}{\underline{\kappa}_{j+1}}\}$

Choose

$$K = \frac{1}{\min\{\gamma_{\text{ev}}, \gamma_{\text{ess}}\}}$$

(A3)/(A4) Non-decreasing function g

Need a non-decreasing function $g\colon [0, \infty) \to [0, \infty)$ which satisfies

$$\left\| F'(\tilde{u} + u) - F'\tilde{u} \right\|_{\mathcal{B}} \leq g(\|u\|_{\mathrm{H}^1}) \quad (u \in \mathrm{H}^1_S(\mathbb{R})) \tag{A3}$$

and $g(t) \to 0 \; (t \to 0^+)$ (A4)

(A3)/(A4) Non-decreasing function g

Need a non-decreasing function $g\colon [0,\infty) \to [0,\infty)$ which satisfies

$$\left|\left(F'(\tilde{u}+u)v - (F'\tilde{u})v\right)[\varphi]\right| \leq g\left(\|u\|_{H^1}\right)\|v\|_{H^1}\|\varphi\|_{H^1} \quad \left(u,v,\varphi \in H^1_S(\mathbb{R})\right) \qquad \text{(A3)}$$

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(A3)/(A4) Non-decreasing function g

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and $g(t) \to 0 \ (t \to 0^+)$ (A4)

Set

$$g(t) = \frac{3t}{2\sigma^{\frac{3}{2}}}\left(2\|\tilde{u}\|_{H^1} + t + \frac{1}{\sqrt{c}}\left(2\|\tilde{u}\|_{L^2} + \frac{t}{\sqrt{\sigma}}\right)\right)$$

Results for the one-dimensional system

Proved a non-trivial solution in the following cases:

c	V_0	σ	δ	K	α
30.0	1.0	2.133	3.085e-4	3.753	1.17e-3
40.0	1.0	1.973	3.154e-4	3.543	1.12e-3
50.0	1.0	1.866	3.174e-4	3.498	1.12e-3

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