TECHNISCHE UNIVERSITÄT DRESDEN

DRESDEN concept

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Static Output Feedback Control by Interval Eigenvalue Placement using Quantifier Elimination

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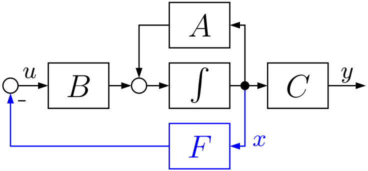
Motivation

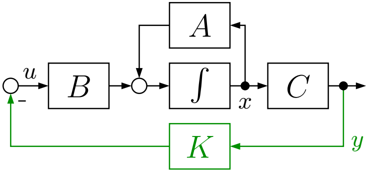
Linear time-invariant state-space system

$$\dot{x} = Ax + Bu, \quad y = Cx$$

State feedback



Static output feedback



- ⊕ simple calculation of the gain
- ⊖ observer required
- ⊖ dynamic output feedback

- ⊖ difficult existence / calculation
- ⊕ no controller dynamics
- ⊕ simple implementation

Structure of the Talk

State Feedback Design

State Feedback Design
Eigenvalue Assignment

State-space system

$$\dot{x} = Ax + Bu, \quad A \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{n\times m}$$

Static state feedback

$$u = -Fx, \quad F \in \mathbb{R}^{m\times n}$$

Closed-loop system

$$\dot{x} = (A - BF)\,x$$

Eigenvalues of the closed-loop system can be placed arbitrarily if and only if the system is controllable, i.e.,

$$\forall s \in \mathbb{C}: \quad \operatorname{rank}\,(sI - A, B) = n$$

Explicit design procedures: Ackermann's formula, ...

State Feedback Design

Stabilization

The system is called stabilizable if

$$\exists F : \quad A - BF \text{ is Hurwitz}$$

The system is stabilizable if and only if

$$\forall s \in \mathbb{C}, \Re(s) \geq 0 : \quad \operatorname{rank}(sI - A, B) = n$$

Stability test with a Lyapunov candidate function $V(x) = x^T P x$, $P \succ 0$

$$\begin{aligned} \dot{V}(x) &= \dot{x}^T P x + x^T P \dot{x} \\ &= x^T (A - BF)^T P x + x^T P (A - BF) x \\ &= -x^T Q x \quad \text{with} \quad Q \succ 0 \end{aligned}$$

Lyapunov equation

$$A^T P + PA - F^T B^T P - PBF = -Q$$

State Feedback Design
Stabilization

Lyapunov equation

$$A^T P + PA - F^T B^T P - PBF = -Q$$

Bilinear matrix inequality

$$A^T P + PA - F^T B^T P - PBF \prec 0$$

Multiplying both sides with $W := P^{-1}$ yields

$$WA^T + AW - WF^T B^T - BFW \prec 0$$

With $Y = FW$ ($\Leftrightarrow F = YW^{-1}$): Linear matrix inequality (LMI)

$$WA^T + AW - Y^T B^T - BY \prec 0$$

Static Output Feedback Design

Static Output Feedback Design
Eigenvalue Assignment

State-space system

$$\dot{x} = Ax + Bu, \quad y = Cx, \quad C \in \mathbb{R}^{r \times n}$$

Static state feedback

$$u = -Ky, \quad K \in \mathbb{R}^{m \times r}$$

Closed-loop system

$$\dot{x} = (A - BKC)\, x$$

Necessary condition for arbitrary eigenvalue asignment (Herman and Martin 1977):

$$mr \geq n$$

Sufficient condition: generic case (Davison and Wang 1975, Wang 1992), open problem (Rosenthal and Willems 1999), solved (Franke 2014)

Static Output Feedback Design
Stabilization

The system is called stabilizable by static output feedback if

$$\exists K : \quad A - BKC \text{ is Hurwitz}$$

Lyapunov candidate function $V(x) = x^T P x$, $P \succ 0$ yields

$$P(A + BKC) + (A + BKC)^T P \prec 0$$

Alternative formulation with $W := P^{-1}$

$$(A + BKC)W + W(A + BKC)^T \prec 0$$

Difficult to solve simultaneously in (P, K) or (W, K) due to bilinearity!

Static Output Feedback Design
Why is it difficult?

- Consider an $n \times n$ matrix $M = (m_{ij})$. The determinant

$$\det M = \sum_{\pi} \operatorname{sign}(\pi)\, m_{i\pi(i)} \cdots m_{n\pi(n)}$$

  is multilinear in the entries m_{ij}
- The coefficients $a_0, \ldots, a_{n-1}$ of the characteristic polynomial

$$\det(sI - M)$$

  are multilinear in the entries m_{ij}
- The assignment problem

$$\mathrm{CP}(s) = \det(sI - (A - BKC)) \stackrel{!}{=} \mathrm{CP}^*(s)$$

  with a designed characteristic polynomial $\mathrm{CP}^*(s)$ is multilinear in the entries of the gain matrix $K = (k_{ij})$.

Static Output Feedback Design
Number of Solutions

The eigenvalue assignment problem multilinear.

Schubert Number

Numer of complex solutions for a *generic* system:

$$d(m,r) = \frac{1!\,2!\,\cdots(r-1)!\,1!\,2!\cdots(m-1)!\,(mr)!}{1!\,2!\cdots(m+r-1)!}$$

If $d(m,r)$ is odd $\implies$ there exists a real solution K
for the generic system (Schumacher 1980)

Static Output Feedback Design
Some Desicion Problems

Problem (Arbitrary Eigenvalue Placement)

Can any characteristic polynomial CP^* *be assigned to the closed-loop system with an appropriate matrix* K*?*

Problem (Specific Eigenvalue Placement)

Consider a given characteristic polynomial CP^**. Exists a gain matrix* K *such that this polynomial can be assigned to the closed-loop system?*

Problem (Stabilizability)

Exists a gain matrix K *such that all eigenvalues of the closed-loop system have a negative real part?*

Quantifier Elimination

Quantifier Elimination
Basic Definitions

atomic formula: $f(x_1, \cdots, x_k)\ \tau\ 0$ with $\tau \in \{=, <, >, \leq, \geq, \neq\}$ and a polynomial $f(x_1, \ldots, x_k)$

quantifier-free formula: $F(X)$ is a boolean combination of atomic formulas using the operators $\vee, \wedge$ and $\neg$

prenex formula: $G(X, Y) := (Q_1 y_1) \cdots (Q_l y_l)\, F(X, Y)$, with $Q_i \in \{\exists, \forall\}$ and the quantifier-free polynomial $F(X, Y)$

quantified variables: $y_1, \ldots, y_l$

free variables: $x_1, \ldots, x_k$

Quantifier Elimination
Over The Real Closed Field

Direct consequence of the Tarski-Seidenberg Theorem
(Tarksi 1948, Seidenberg 1954):

Theorem (Quantifier Elimination Over The Real Closed Field)

For every real prenex formula $G(X, Y)$ exists an equivalent quantifier-free formula $H(X)$.

For every $Y \in \mathbb{R}^l$ and $X \in \mathbb{R}^k$:

$$G(X, Y) \text{ is } \texttt{true} \quad \Longleftrightarrow \quad H(X) \text{ is } \texttt{true}$$

Quantifier Elimination
Small Example

Consider the quadratic polynomial $f(x) = x^2 + px + q$

- Existence of a real zero

$$\exists x : f(x) = 0 \iff p^2 - 4q \geq 0$$

- Existence of two different real zeros

$$\exists x, y : f(x) = 0 \wedge f(y) = 0 \wedge x \neq y \iff p^2 - 4q > 0$$

- Is $f(x)$ positive?

$$\forall x : f(x) > 0 \iff p^2 - 4q < 0$$

- Can be $f(x)$ negative?

$$\exists p, q \, \forall x : f(x) < 0 \iff \texttt{false}$$

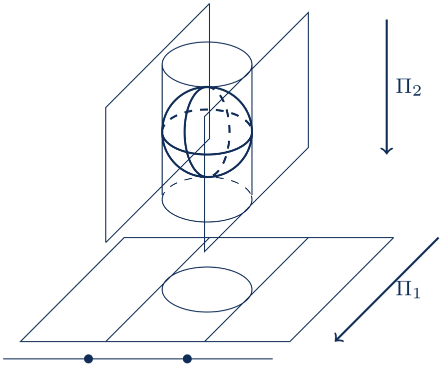
Quantifier Elimination
Common Algorithms

- Cylindrical Algebraic Decomposition (CAD)
  - For arbitrary polynomials
  - $\mathcal{O}(2^{2^i})$, i: number of variables (worst case)
- Virtual Substitution (VS)
  - For linear, quadratic and cubic polynomial w.r.t. the quantified variables
  - $\mathcal{O}(2^j)$, j: number of quantified variables
- Sturm-Habicht-Sequences (SH) and Real Root Classification (RRC) (Sign Definite Condition, SDC)
  - $\forall x : f_1(x) > 0$, $\exists x : f_2(x) = 0$, $\exists x : f_3(x) < 0$
  - $\forall x : x \geq 0 \implies f(x) > 0$, $\mathcal{O}(2^k), k = \deg_x(f(x))$

Quantifier Elimination

Cylindrical Algebraic Decomposition (CAD)

- Decomposition into semi-algebraic sets (cells)
- Signs of function values of all polynomials in the cell are constant
- Cylindrical:
  - Π_k is projection from $\mathbb{R}^n$ to $\mathbb{R}^{n-k}$ with $1 \leq k < n$
  - For every pairs a und b of cells holds: Either $\Pi(a) = \Pi(b)$ or $\Pi(a) \cap \Pi(b) = \varnothing$

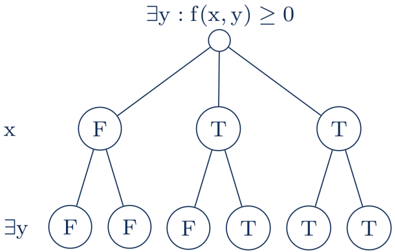


Quantifier Elimination

Cylindrical Algebraic Decomposition (CAD)

CAD algorthims: 1. Projection
2. Classification in $\mathbb{R}^1$
3. Lifting back to $\mathbb{R}^2 \ldots \mathbb{R}^n$

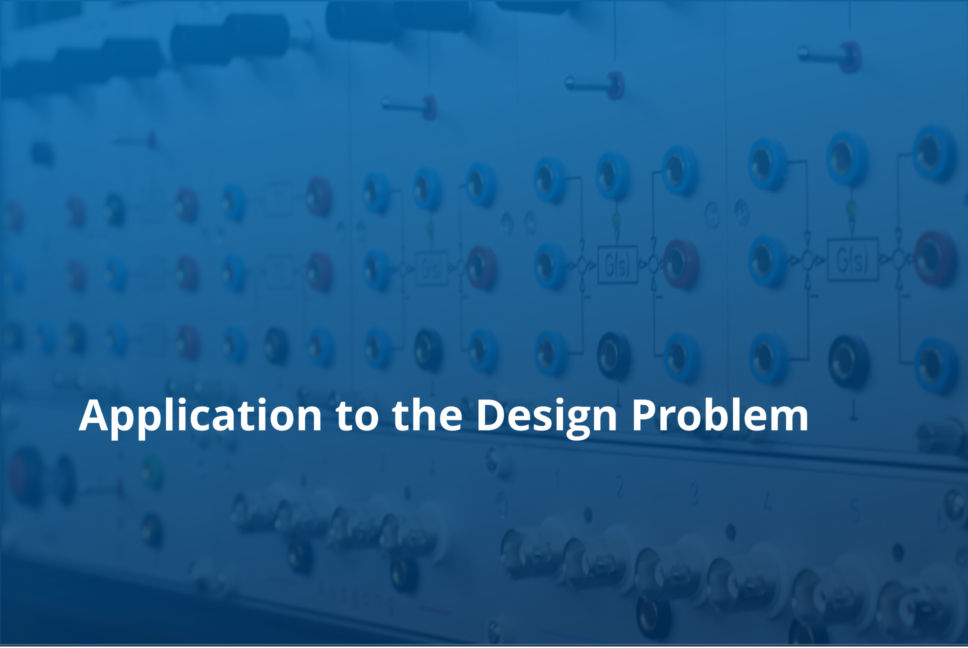
The resulting cells descriptions can be organized in a tree:



Quantifier Elimination

Quantifier Elimination Software

QEPCAD	CAD	Quantifier Elimination by Partial Cylindrical Algebraic Decomposition *Collins, Hong 1991*
QEPCAD B	CAD	*Brown 2003*
Redlog	CAD, VS	Reduce package *Dolzmann, Sturm 1997*
SyNRAC	SDC	Maple toolbox *Anai, Yanami 2003*
RegularChains	CAD	Maple toolbox *Chen, Maza 2014*



Application to the Design Problem

Application to the Design Problem

Arbitrary Eigenvalue Placement

Closed-loop characteristic polynomial

$$\mathrm{CP}(s) = a_0 + a_1 s + \cdots + a_{n-1} s^{n-1} + s^n$$

Prescribed / desired characteristic polynomial

$$\mathrm{CP}^*(s) = a_0^* + a_1^* s + \cdots + a_{n-1}^* s^{n-1} + s^n$$

Problem (Arbitrary Eigenvalue Placement)

Can any characteristic polynomial CP^* *be assigned to the closed-loop system with an appropriate matrix* K*?*

Formulation with quantifiers:

$$\forall a_0^* \cdots \forall a_{n-1}^* \, \exists k_{11} \cdots \exists k_{mp} : \quad a_0 = a_0^* \wedge \ldots \wedge a_{n-1} = a_{n-1}^*$$

Application to the Design Problem
Formalization

Problem (Specific Eigenvalue Placement)

Consider a given characteristic polynomial CP^*. *Exists a gain matrix* K *such that this polynomial can be assigned to the closed-loop system?*

Formulation with quantifiers:

$$\exists k_{11} \cdots \exists k_{mr} : \quad a_0 = a_0^* \wedge \ldots \wedge a_{n-1} = a_{n-1}^*$$

Computation of the gain matrix $K = (k_{ij})$:

1. omit existence quantifier for one variable k_{ij}
2. compute the feasible set
3. specify the variable and proceed with the next variable

Application to the Design Problem
Formalization

Problem (Stabilizability)

Exists a gain matrix K such that all eigenvalues of the closed-loop system have a negative real part?

Use existence quantifiers $\exists k_{11} \cdots \exists k_{mr}$ in connection with Routh, Hurwitz, Liénard-Chipart test:

$$
\begin{aligned}
n &= 2: \quad && a_0 > 0 \wedge a_1 > 0 \\
n &= 3: \quad && a_0 > 0 \wedge a_1 > 0 \wedge a_2 > 0 \wedge a_1 a_2 - a_0 > 0 \\
n &= 4: \quad && a_0 > 0 \wedge a_1 > 0 \wedge a_2 > 0 \wedge a_3 > 0 \wedge \\
& && \ldots \\
&\vdots
\end{aligned}
$$

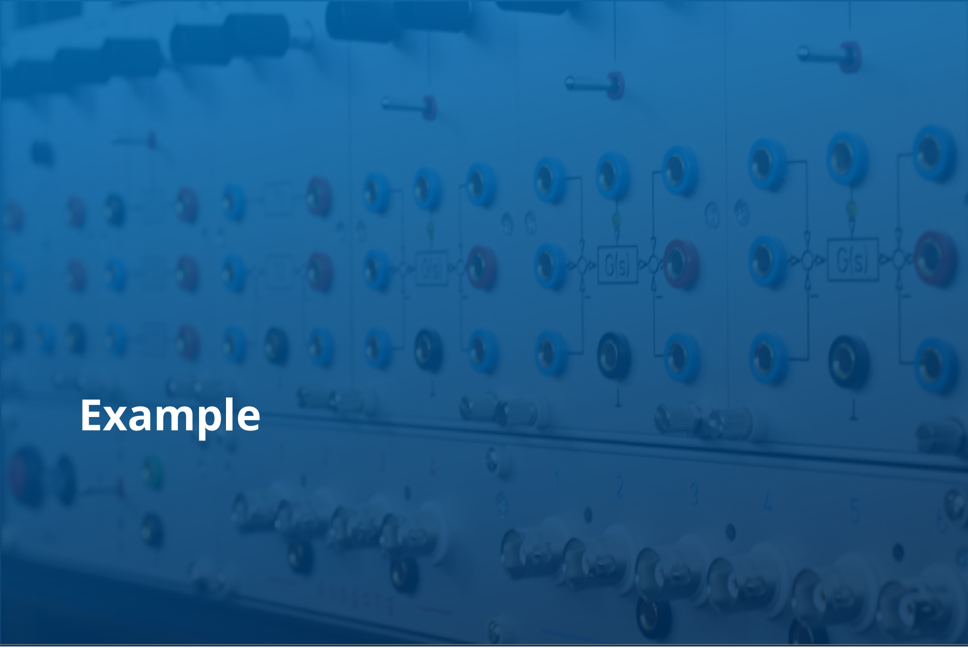
Application to the Design Problem
Formalization

Real eigenvalue assignment: No oscillations

$$\mathrm{CP}^*(s) = (s - s_1) \cdots (s - s_n)$$

Possible design goals:

- real stabilization: $s_1 < 0, \ldots, s_n < 0$
- robust real stabilization: $s_1 \leq s_0, \ldots, s_n \leq s_0$ with $s_0 < 0$
- interval assignment: $s_1 \in \mathcal{I}_1, \ldots, s_n \in \mathcal{I}_n$
- any combination thereof



Example

Example
Anderson, Bose, Jury 1975

System

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 13 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & -5 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} k_1 & k_2 \end{pmatrix}$$

Closed-loop characteristic polynomial

$$\begin{aligned} \mathrm{CP}(s) &= \det(sI - (A + BKC)) \\ &= a_0 + a_1 s + a_2 s^2 + s^3 \\ &= k_2 + (k_2 - 5k_1 - 13)s + k_1 s^2 + s^3. \end{aligned}$$

Example
Stabilizability

Stability condition: The Liénard-Chipart test

$$a_0 > 0 \wedge a_2 > 0 \wedge a_1 a_2 - a_0 > 0$$

yields

$$k_1 > 0 \wedge k_2 > 0 \wedge k_1 k_2 - k_2 - 5k_1^2 - 13k_1 > 0. \tag{1}$$

Question

Does to the set of nonlinear inequalities (1) have a real solution?

Formulation as the prenex formula

$$\exists k_1, k_2 : \quad \text{Cond. (1)}$$

yields `true` $\Longrightarrow$ stabilization problem solvable!

Example
Computation fo the gain matrix

- *Anderson et al 1975* (by hand), *Syrmos et al 1997* (with QEPCAD B): Using k_1 as a free variable

$$\exists k_2 : \quad \text{Cond. (1)}$$

  results in $k_1 > 1$. With $k_1 := 2 \implies k_2 > 46$, e.g. $k_2 := 50$

- *Alternative computation* (with Redlog): Using k_2 as a free variable

$$\exists k_1 : \quad \text{Cond. (1)}$$

  results in $k_2 > 13 \wedge k_2^2 - 46k_2 + 169 > 0$. Largest real root: $k_2 = 23 + 6\sqrt{10} \approx 41.97$. With $k_2 := 42 \implies 2.8 < k_1 < 3$, e.g. $k_1 = 2.9$

Example
Eigenvalue Assignment

- Arbitrary eigenvalue assignment: Not possible ($2 = mr < n = 3$)
- Specific eigenvalue assignment:

$$s_1 = s_2 = -3,\ s_3 = -5 \quad \Longrightarrow \quad \begin{aligned} \mathrm{CP}^*(s) &= a_0^* + a_1^* a + a_2^* s^2 + s^3 \\ &= 45 + 39s + 11s^2 + s^3 \end{aligned}$$

Decision problem as prenex formula

$$\exists k_1, k_2 : \quad a_0 = a_0^* \wedge a_1 = a_1^* \wedge a_2 = a_2^*$$

yields `false`

Example
Eigenvalue Assignment

- Real interval assigment: $\mathcal{I}_1 = \mathcal{I}_2 = [-5, -1]$, $\mathcal{I}_3 = [-10, -1]$
  Decision problem as prenex formula

$$\begin{aligned}\exists s_1, s_2, s_3, k_1, k_2: \quad & -5 \leq s_1 \leq -1 \wedge \\ & -5 \leq s_2 \leq -1 \wedge \\ & -10 \leq s_3 \leq -1 \wedge \\ & a_0 = a_0^* \wedge a_1 = a_1^* \wedge a_2 = a_2^*\end{aligned}$$

  yields `true`

Summary

Summary

- Formulation of different specifications of eigenvalue assignment and stabilization as decision problems
- Existence and calculation of the gain matrix using quantifier elimination
- Exact formulation of possible degrees of freedom
- Restrictions in applicability due to computational complexity, but now applicable to non-trivial systems
- Virtual substitution method is significantly faster than cylindrical algebraic decomposition

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