Observability of Nonlinear Systems Using Interval Analysis

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Motivation

Why deal with observability?

- To assure the functionality of an observer first the observability has to be proven
- Many applications have a finite state space, therefore only observability for this finite state space is of interest
- In the case of a system which is not observable, it can be of interest which states can not be distinguished of each other

Nonlinear autonomous Systems

$$\dot{x}(t) = f(x(t)), \quad x(0) = x_0$$

$$y(t) = h(x(t))$$

with $f : \mathbb{R}^n \to \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}^m$ as real-analytic functions

Problem formulation

Lie derivatives

$$y(t) = \sum_{k \geq 0} \frac{t^k}{k!} \left(L_f^k h \right) (x_0)$$

$$L_f^k h = \frac{\partial}{\partial x} \left(L_f^{k-1} h \right) f \text{ and } L_f^0 h = h$$

Distinguishability

$$z^1 \text{ and } z^2 \text{ are indistinguishable} \Leftrightarrow y^1(t) \equiv y^2(t)$$

$$\Leftrightarrow \left(L_f^k h \right) (z^1) = \left(L_f^k h \right) (z^2), k \geq 0$$

$$\mathcal{M} = \left\{ \begin{pmatrix} z^1 \\ z^2 \end{pmatrix} \middle| \left(L_f^k h \right) (z^1) = \left(L_f^k h \right) (z^2) \right\} / \left\{ \begin{pmatrix} z^1 \\ z^2 \end{pmatrix} \middle| z^1 = z^2 \right\} = \emptyset$$

Problem formulation

Observability function

$$q\left(x\right)=\begin{bmatrix} L_f^0 h\left(x\right) \\ \vdots \\ L_f^\kappa h\left(x\right) \end{bmatrix}$$

with κ as the necessary amount of Lie derivatives.
z^1 and z^2 are indistinguishable $\Leftrightarrow q\left(z^1\right)=q\left(z^2\right)$

Local observability

$$\text{rank}\left(\frac{\partial q\left(x\right)}{\partial x}\right)=n$$

Lie derivatives symbolically

Example nonlinear system

$$f(x) = \begin{pmatrix} -x_2 + x_1\left(1 - x_1^2 - x_2^2\right) \\ x_1 + x_2\left(1 - x_1^2 - x_2^2\right) \end{pmatrix}$$
$$h(x) = x_1 + x_2$$

Lie derivative $L_f^4 h(x)$

$$\begin{aligned}
L_f^4 h(x) = \; & (x_2 + x_1(x_1^2 + x_2^2 - 1))((x_2 + x_1(x_1^2 + x_2^2 - 1))(6x_2 - 2x_1 + 2(2x_1 + 2x_2)(2x_1x_2 - 1) + 2x_2(2x_1x_2 + x_1^2 + 3x_2^2) + 6x_1(2x_1x_2 + 3x_1^2 + x_2^2 - 2) \\
& +2(6x_1 + 2x_2)(3x_1^2 + x_2^2 - 1) + 6x_1(x_1^2 + x_2^2 - 1) + 2x_2(x_1^2 + x_2^2 - 1)) - (x_1 - x_2(x_1^2 + x_2^2 - 1))(2x_2 - 2x_1 + (2x_1 + 6x_2)(2x_1x_2 - 1) + (6x_1 + 2x_2)(2x_1x_2 + 1) \\
& +2x_1(2x_1x_2 + x_1^2 + 3x_2^2) + 2x_2(2x_1x_2 + 3x_1^2 + x_2^2 - 2) + (2x_1 + 2x_2)(3x_1^2 + x_2^2 - 1) + (2x_1 + 2x_2)(x_1^2 + 3x_2^2 - 1) + 2x_1(x_1^2 + x_2^2 - 1) + 2x_2(x_1^2 + x_2^2 - 1)) \\
& +(2x_1x_2 - 1)((2x_1 + 2x_2)(x_2 + x_1(x_1^2 + x_2^2 - 1)) - (2x_1 + 6x_2)(x_1 - x_2(x_1^2 + x_2^2 - 1)) + (x_1^2 + 3x_2^2 - 1)(2x_1x_2 + x_1^2 + 3x_2^2) + (2x_1x_2 + 1)(2x_1x_2 + 3x_1^2 + x_2^2 - 2)) \\
& +(3x_1^2 + x_2^2 - 1)((6x_1 + 2x_2)(x_2 + x_1(x_1^2 + x_2^2 - 1)) - (2x_1 + 2x_2)(x_1 - x_2(x_1^2 + x_2^2 - 1)) + (3x_1^2 + x_2^2 - 1)(2x_1x_2 + 3x_1^2 + x_2^2 - 2) + (2x_1x_2 - 1)(2x_1x_2 + x_1^2 + 3x_2^2))) \\
& -(x_1 - x_2(x_1^2 + x_2^2 - 1))((x_2 + x_1(x_1^2 + x_2^2 - 1))(2x_2 - 2x_1 + (2x_1 + 6x_2)(2x_1x_2 - 1) + (6x_1 + 2x_2)(2x_1x_2 + 1) + 2x_1(2x_1x_2 + x_1^2 + 3x_2^2) + 2x_2(2x_1x_2 + 3x_1^2 + x_2^2 - 2) \\
& +(2x_1 + 2x_2)(3x_1^2 + x_2^2 - 1) + (2x_1 + 2x_2)(x_1^2 + 3x_2^2 - 1) + 2x_1(x_1^2 + x_2^2 - 1) + 2x_2(x_1^2 + x_2^2 - 1)) - (x_1 - x_2(x_1^2 + x_2^2 - 1))(2x_2 - 6x_1 + 2(2x_1 + 2x_2)(2x_1x_2 + 1) \\
& +6x_2(2x_1x_2 + x_1^2 + 3x_2^2) + 2x_1(2x_1x_2 + 3x_1^2 + x_2^2 - 2) + 2(2x_1 + 6x_2)(x_1^2 + 3x_2^2 - 1) + 2x_1(x_1^2 + x_2^2 - 1) + 6x_2(x_1^2 + x_2^2 - 1)) + (2x_1x_2 + 1)((6x_1 + 2x_2)(x_2 \\
& +x_1(x_1^2 + x_2^2 - 1)) - (2x_1 + 2x_2)(x_1 - x_2(x_1^2 + x_2^2 - 1)) + (3x_1^2 + x_2^2 - 1)(2x_1x_2 + 3x_1^2 + x_2^2 - 2) + (2x_1x_2 - 1)(2x_1x_2 + x_1^2 + 3x_2^2)) + (x_1^2 + 3x_2^2 - 1)((2x_1 + 2x_2) \\
& (x_2 + x_1(x_1^2 + x_2^2 - 1)) - (2x_1 + 6x_2)(x_1 - x_2(x_1^2 + x_2^2 - 1)) + (x_1^2 + 3x_2^2 - 1)(2x_1x_2 + x_1^2 + 3x_2^2) + (2x_1x_2 + 1)(2x_1x_2 + 3x_1^2 + x_2^2 - 2)))
\end{aligned}$$

Using power series for Lie derivatives

Power series

$$a(t) := \sum_{k=0}^{\infty} a_k \; t^k$$

$(a \pm b)(t)$	$\sum_{k=0}^{\infty} (a_k \pm b_k)\, t^k$
$(a \cdot b)(t)$	$\sum_{k=0}^{\infty} \left(\sum_{l=0}^{k} a_l \cdot b_{k-l} \right) t^k$
$(a : b)(t)$	$\sum_{k=0}^{\infty} \left(\frac{a_k}{b_0} - \frac{1}{b_0} \sum_{l=0}^{k} c_l \cdot b_{k-l} \right) t^k$

Using power series for Lie derivatives

Computation of $L_f^0 h\left(x_0\right)$

$$x_0 \text{ given} \qquad L_f^0 h\left(x_0\right) = 0!\ h_0\left(x_0\right)$$

Computation of $L_f^1 h\left(x_0\right)$

$$x_1 = \tfrac{1}{1+1} f_0(x_0) \quad L_f^1 h\left(x_0\right) = 1!\ h_1\left(x_0, x_1\right)$$

Computation of $L_f^2 h\left(x_0\right)$

$$x_2 = \tfrac{1}{2+1} f_1(x_0, x_1) \quad L_f^2 h\left(x_0\right) = 2!\ h_2\left(x_0, x_1, x_2\right)$$

Computation of k Lie derivatives

$$x_{k+1} = \tfrac{1}{k+1} f_k \text{ with } f_k = f_k\left(x_0, \ldots, x_k\right) \text{ and } f\left(x\left(t\right)\right) = \sum_{k \geq 0} f_k\ t^k$$

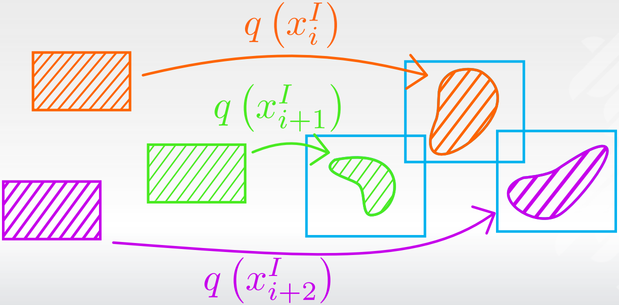
$$L_f^k h\left(x_0\right) = k!\ h_k\left(x_0, \ldots, x_k\right)$$

Notation

Notations in this talk

Interval	$x^I = [\underline{x}, \overline{x}]$
Interval midpoint	$\check{x} = \frac{\underline{x}+\overline{x}}{2}$
Interval radius	$\hat{x} = \frac{\overline{x}-\underline{x}}{2}$
Hull of intervals	$\tilde{x}^I = \cup\ x_i^I$ with $i = 1, ..., n$

List structure



Double layered list structure

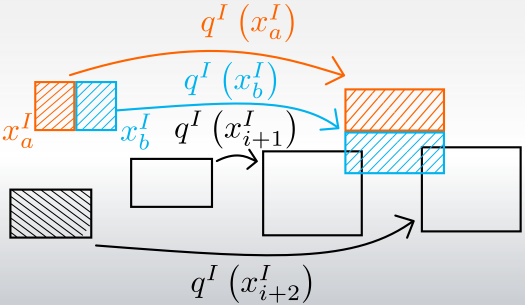
$$\mathcal{L}_i = \left\{ x_i^I \mid \Theta^I\left(x_i^I\right) \right\}, \quad \Theta^I\left(x_i^I\right) = x_{i+1}^I, x_{i+2}^I$$

$$\mathcal{L}_{i+1} = \left\{ x_{i+1}^I \mid \Theta^I\left(x_{i+1}^I\right) \right\}, \quad \Theta^I\left(x_{i+1}^I\right) = x_i^I$$

$$\mathcal{R} = \left\{ \mathcal{L}_i, \mathcal{L}_{i+1}, \mathcal{L}_{i+2}, \ldots \right\}$$

Distinguish the mapping $q^I(x^I)$

$$\mathcal{L}_i = \left\{x_i^I \mid x_{i+1}^I, x_{i+2}^I\right\}$$



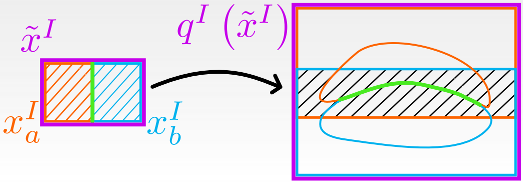
Distinguishable if $q\left(x_1^I\right) \cap q\left(x_2^I\right) = \emptyset$

$$\mathcal{L}_a = \left\{x_a^I \mid x_b^I\right\}$$

$$\mathcal{L}_b = \left\{x_b^I \mid x_a^I, x_{i+1}^I, x_{i+2}^I\right\}$$

take first $\mathcal{L}_i$ out of $\mathcal{R}$
bisect $\mathcal{L}_i \ni x_i^I = x_a^I \cup x_b^I$ and remove x_i^I entirely from $\mathcal{R}$
calculate $q\left(x_a^I\right)$ and $q\left(x_b^I\right)$
test Distinguishability and create $\mathcal{L}_a$ and $\mathcal{L}_b$
for $j = a, b$ **do**
    get hull $\tilde{x}_j^I$ for $\mathcal{L}_j$
    if $\tilde{x}_j^I$ local observable **then**
        remove x_j^I from $\mathcal{R}$
    else
        put $\mathcal{L}_j$ at the end of $\mathcal{R}$
    end if
end for
change κ if necessary

Problem of the Distinguishability



$$\Rightarrow q\left(x_a^I\right) \cap q\left(x_b^I\right) \neq \emptyset$$

Therefore we need the local observability criterion

$$\text{rank}\left(\frac{\partial q^I\left(\tilde{x}^I\right)}{\partial x}\right) = n$$

take first $\mathcal{L}_i$ out of $\mathcal{R}$
bisect $\mathcal{L}_i \ni x_i^I = x_a^I \cup x_b^I$ and remove x_i^I entirely from $\mathcal{R}$
calculate $q\left(x_a^I\right)$ and $q\left(x_b^I\right)$
test Distinguishability and create $\mathcal{L}_a$ and $\mathcal{L}_b$
for $j = a, b$ **do**
    get hull $\tilde{x}_j^I$ for $\mathcal{L}_j$
    if $\tilde{x}_j^I$ local observable **then**
        remove x_j^I from $\mathcal{R}$
    else
        put $\mathcal{L}_j$ at the end of $\mathcal{R}$
    end if
end for
change κ if necessary

Local observability criterion

With J^I been the Jacobian of $q^I\left(\tilde{x}^I\right)$

$$V^I := \begin{pmatrix} 0 & J^I \\ {J^I}^\intercal & 0 \end{pmatrix}$$

Theorem of Rohn [98]:

$$\underline{\lambda}_i^I\left(V^I\right) = \lambda_i\left(\check{V}\right) - \rho\left(\hat{V}\right)$$

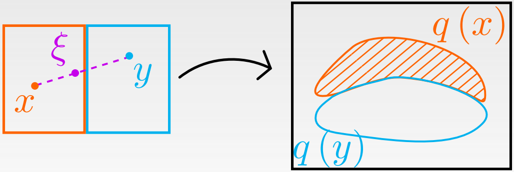
$$\overline{\lambda}_i^I\left(V^I\right) = \lambda_i\left(\check{V}\right) + \rho\left(\hat{V}\right)$$

$\rho\left(\hat{V}\right)$: spectral radius of $\hat{V}$

If $2n\ \lambda_i^I \not\ni 0 \Rightarrow V^I$ is of full rank and therefore J^I is of full rank

take first $\mathcal{L}_i$ out of $\mathcal{R}$
bisect $\mathcal{L}_i \ni x_i^I = x_a^I \cup x_b^I$ and remove x_i^I entirely from $\mathcal{R}$
calculate $q\left(x_a^I\right)$ and $q\left(x_b^I\right)$
test Distinguishability and create $\mathcal{L}_a$ and $\mathcal{L}_b$
for $j = a, b$ **do**
    get hull $\tilde{x}_j^I$ for $\mathcal{L}_j$
    if $\tilde{x}_j^I$ local observable **then**
        remove x_j^I from $\mathcal{R}$
    else
        put $\mathcal{L}_j$ at the end of $\mathcal{R}$
    end if
end for
change κ if necessary

Local observability criterion



$$q_i(x) - q_i(y) = \left(\frac{\partial q_i(\xi_i)}{\partial x}\right)^{\mathsf{T}} (x - y)$$

$$q(x) - q(y) = \underbrace{\begin{bmatrix} \left(\frac{\partial q_i(\xi_1)}{\partial x}\right)^{\mathsf{T}} \\ \left(\frac{\partial q_i(\xi_2)}{\partial x}\right)^{\mathsf{T}} \\ \vdots \end{bmatrix}}_{M} (x - y)$$

$$M \in M^I = \frac{\partial q^I(x^I)}{\partial x}$$

```
take first L_i out of R
bisect L_i ∋ x_i^I = x_a^I ∪ x_b^I and
  remove x_i^I entirely from R
calculate q(x_a^I) and q(x_b^I)
test Distinguishability
  and create L_a and L_b
for j = a, b do
    get hull x̃_j^I for L_j
    if x̃_j^I local observable then
        remove x_j^I from R
    else
        put L_j at the end of R
    end if
end for
change κ if necessary
```

Determination the amount of Lie derivatives

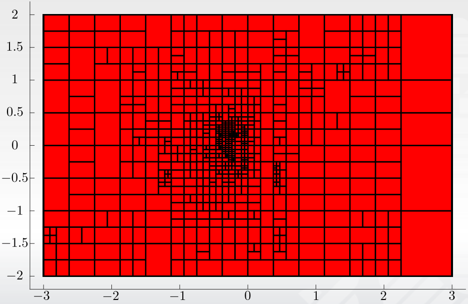
- Since $h : \mathbb{R}^n \to \mathbb{R}^m$ every further Lie derivative adds m equations to $q(x)$
- Necessary amount of Lie derivatives vary due to the observability singularities
- At least n equations are mandatory
- Always computing $\kappa + 1$ Lie derivatives
- Test Distinguishability and local observability for $\kappa - 1$, κ and $\kappa + 1$

take first $\mathcal{L}_i$ out of $\mathcal{R}$
bisect $\mathcal{L}_i \ni x_i^I = x_a^I \cup x_b^I$ and remove x_i^I entirely from $\mathcal{R}$
calculate $q\left(x_a^I\right)$ and $q\left(x_b^I\right)$
test Distinguishability and create $\mathcal{L}_a$ and $\mathcal{L}_b$
for $j = a, b$ **do**
    get hull $\tilde{x}_j^I$ for $\mathcal{L}_j$
    if $\tilde{x}_j^I$ local observable **then**
        remove x_j^I from $\mathcal{R}$
    else
        put $\mathcal{L}_j$ at the end of $\mathcal{R}$
    end if
end for
change κ if necessary

Nonlinear example

Nonlinear state space representation

$$\begin{aligned} f(x) &= \begin{pmatrix} x_2 \\ -x_1 \end{pmatrix} \\ h(x) &= x_1^2 + x_1 \\ L_f^0 h(x) &= x_1^2 + x_1 \\ L_f^1 h(x) &= 2x_1x_2 + x_2 \end{aligned}$$



Solutions for $L_f^0 h(x) = L_f^0 h(z)$ and $L_f^1 h(x) = L_f^1 h(z)$ are $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ is indistinguishable from $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ and $\begin{pmatrix} -z_1 - 1 \\ -z_2 \end{pmatrix}$

Further Lie derivatives are necessary to exclude the second solution

$$\begin{aligned} L_f^2 h(x) &= 2x_2^2 - 2x_1^2 - x_1 \\ L_f^3 h(x) &= -8x_1x_2 - x_2 \end{aligned}$$

What was shown

- Numeric approach to determine whether a nonlinear system is observable
- The presented algorithm can handle nonlinear systems with real-analytic functions
- The presented algorithm can adjust the necessary amount of Lie derivatives dynamically
- In case of a not observable system the algorithm is capable of finding the states that can not be distinguished of each other

What we currently work on

- Reduction of wrapping effect
- Improving the runtime further by parallelizing the algorithm

Thank you for your attention
Any questions?