Inner and outer approximation of the Viability Kernel for a bounded uncertai system

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Introduction:Stability

- Stability is the property of a system (mechanical, chemical or philosophycal)to not self-destruct [1].
- Stability theory for mechanical systems addresses the problem of verifying that the energy of the system always decreases. Different methods such as Lyapuov theory are well developed for this purpose.

Introduction:Stability

- In control theory, stability analysis is a very important area of research that verifies if for a given input the evolution of a system reaches a certain region of the state space or if it reaches a forbiden region.
- To this end, Viability theory is used to compute a region where the evolutions of the system will remain indefinetly.

Introduction:Viability

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Consider a dynamical system $\mathcal{G}$ defined by

System

$$\dot{x}(t) = f(x(t), u(t))$$
$$u(t) \in \mathbb{U}$$

where $x(t) \in \mathbb{R}^n$, $\mathbb{U}$ is a compact subset of $\mathbb{R}^m$, $u \in \mathcal{U} = u : \mathbb{R}^+ \longmapsto \mathbb{U}$ and $f : \mathbb{R}^n \times \mathbb{U} \longmapsto \mathbb{R}^n$ to be f a continuous and locally Lipschitzian function bounded in $\mathbb{R}^n \times \mathbb{U}$.

Introduction:Viability

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Let φ to be the flow map of $\mathcal{G}$ that computes the reached state $\varphi(t, x_0, u)$ given an initial state $x_0 = x(t)$ and a control function $u(t)$.

We want to know if there exists at least one evolution of the system $\mathcal{G}$ will remain in the region $\mathbb{K} \subset \mathbb{R}^n, \quad \forall t > 0$ (in other words, if the system is *viable*).

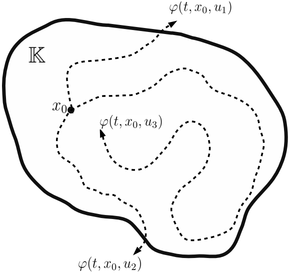

Figure: Example of a Viable set $\mathbb{K}$ for a given system $\mathcal{G}$ [2].

Introduction:Viability Kernel

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If the closed subset $\mathbb{K}$ is not a viability domain, its still possible to look for a closed subsets of $\mathbb{K}$ wich are viable domains; the largest viable subset is called the *Viability Kernel* i.e. $Viab_{\mathcal{S}}(\mathbb{K})$.
The viability kernel can be defined as follows

Viability Kernel

$$Viab\mathcal{G}(\mathbb{K}) = \{x_0 \in \mathbb{K} | \exists u \in \mathcal{U}, \forall t \geq 0, \varphi(t, x, u) \in \mathbb{K}\}. \quad (1)$$

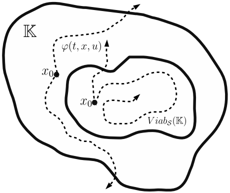

Figure: Example of the Viability Kernel of $\mathbb{K}$ for a given system $\mathcal{G}$ [2].

Viability Kernel

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- In control theory, uncertainties are often added into the models to better describe the behaviour of certain systems.
- For the case of non linear systems, such uncertainties can produce complex dynamics that make the analysis of such systems to be more complicated and impractical using classical methods.
- In such cases, Viability theory offers an alternate view of the problem, in which by using some constrains, guaranteed integration techniques and interval analysis it is possible to obtain numerically some guaranteed regions in the state space to verify if the system is viable.
- A method for computing the inner and outer approximation of the viability kernel for systems with uncertainties is proposed.

Viability Kernel

Consider a dynamic system $\mathcal{S}$ defined by

System

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)) + \gamma \\ u(t) &\in \mathbb{U} \end{aligned} \tag{2}$$

where $x(t) \in \mathbb{R}^n$ is the system state, $\mathbb{U}$ is a compact subset of $\mathbb{R}^m$, $u \in \mathcal{U} = u : \mathbb{R}^+ \longmapsto \mathbb{U}$, $f : \mathbb{R}^n \times \mathbb{U} \longmapsto \mathbb{R}^n$ being f a continuous and locally Lipschitzian function bounded in $\mathbb{R}^n \times \mathbb{U}$, Γ is a compact subset of $\mathbb{R}^p$ and $\gamma \in \Gamma$.

Viability Kernel

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For the system $\mathcal{S}$, an initial set $\mathbb{X}_0$, a time horizon t_H and an input set $\mathbb{U}$; the evolution of the system given by $\varphi([0, t_H], \mathbb{X}_0, \mathbb{U})$ can be bounded by a tube $\mathbb{T}_{\mathcal{S}}([0, t_H], \mathbb{X}_0, \mathbb{U})$ such that

$$\varphi([0, t_H], \mathbb{X}_0, \mathbb{U}) \subset \mathbb{T}_{\mathcal{S}}([0, t_H], \mathbb{X}_0, \mathbb{U}) \tag{3}$$

This tube is a capture tube for the system $\mathcal{S}$; where the border of this tube is denoted by $\partial\mathbb{T}_{\mathcal{S}}([0, t_H], \mathbb{X}_0, \mathbb{U})$.

Viability Kernel

Let $\mathbb{B}_{\mathcal{S}}(t_H, \mathbb{X}_0, \mathbb{U})$ to be the set of states of the capture tube, at time instant t_H, i.e. $\partial\mathbb{T}_{\mathcal{S}}(t_H, \mathbb{X}_0, \mathbb{U})$.
The tube boundary $\partial\mathbb{T}_{\mathcal{S}}([0, t_H], \mathbb{X}_0, \mathbb{U})$ and the set $\mathbb{B}_{\mathcal{S}}(t_H, \mathbb{X}_0, \mathbb{U})$ satisfy the following conditions:

$$\varphi([0, t_H], \mathbb{X}_0, \mathbb{U}) \cap \partial\mathbb{T}_{\mathcal{S}}([0, t_H], \mathbb{X}_0, \mathbb{U}) = \emptyset, \tag{4}$$

$$\varphi([0, t_H], \mathbb{X}_0, \mathbb{U}) \cap \mathbb{B}_{\mathcal{S}}(t_H, \mathbb{X}_0, \mathbb{U}) \neq \emptyset. \tag{5}$$

Viability Kernel

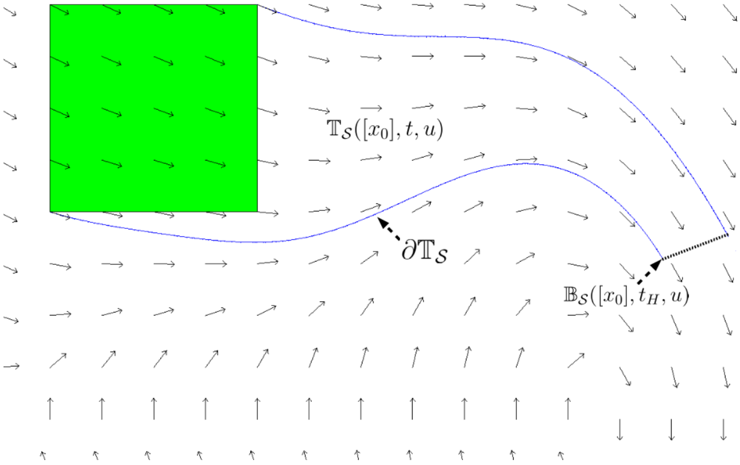


Figure: N-dimensional surface enclosing the evolution of the system.

Viability Kernel

- For this solution, the border of the capture tubes will be used for obtaining an approximation of the viability kernel for systems such as $\mathcal{S}$.
- This approach is an extension of the algorithm proposed by Dominic Monnet, Luc Jaulin and Jordan Ninin, based on interval analysis, to define a guaranteed approximation for the inner and outer regions of the viability kernel.

Viability Kernel

- The algorithm proposed, is based on interval methods, for obtaining the inner ($\mathbb{V}_{in}$) and its complement ($\mathbb{V}_{out}$) approximation of the viability kernel for an initial box $\mathbb{K} \subset \mathbb{R}^n$.
- The implementation of this algorithm, supposes that an initial approximation of the viability kernel has been found. The initial approximation of the viable set, is defined as the boxes of the subpaving $\mathbb{V}_{in}$ that belong to the $Viab\mathcal{S}(\mathbb{K})$. Such set can be obtained using Lyapunov theory or V-viability theory.

Viability Kernel

Computation of an inner and outer approximation of $Viab\mathcal{S}(\mathbb{K})$

- $\forall [x_i] \in \mathbb{H}$, where $\mathbb{H} = \mathbb{K} \setminus (\mathbb{V}_{in} \cup \mathbb{V}_{out})$
- Compute $\partial \mathbb{T}_{\mathcal{S}}([0, t_H], [x_i], [u])$ and $\partial \mathbb{T}_{\mathcal{S}}(t_H, [x_i], [u])$
- Check the following conditions
- $\partial \mathbb{T}_{\mathcal{S}}([0, t_H], [x_i], [u]) \subseteq \mathbb{K}$ and $\partial \mathbb{T}_{\mathcal{S}}(t_H, [x_i], [u]) \subseteq \mathbb{V}_{in} \mapsto \mathbb{V}_{in} := \mathbb{V}_{in} \cup [x_i]$
- $\partial \mathbb{T}_{\mathcal{S}}(t_H, [x_i], [u]) \subseteq ((\mathbb{R}^n \setminus \mathbb{K}) \cup \mathbb{V}_{out}) \mapsto \mathbb{V}_{out} := \mathbb{V}_{out} \cup [x_i]$
- Else bisect $[x]$ and $[u]$

Car on the Hill

- Nonlinear dynamical system where, as the name suggests, a car is being driven on a landscape

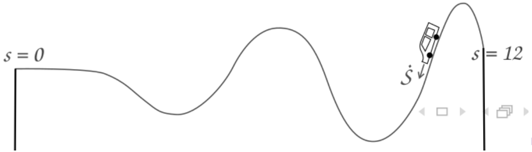
$$g : s \mapsto \frac{\frac{-1.1}{1.2} cos(1.2s) + \frac{1.2}{1.1} cos(1.1s)}{2} \tag{6}$$

where for this case $s \in [0, 12]$.

$$\dot{x}_1(t) = x_2(t) + \gamma, \tag{7}$$

$$\dot{x}_2(t) = -9.81 sin(\frac{\partial g}{\partial x_1}(x_1(t))) - 0.7x_2(t) + u(t) + \gamma \tag{8}$$

- Where $x_1 = s$ is the position of the car, $x_2 = \dot{s}$ is the velocity of the car and $u \in [-2, 2]$ is the control function and $\gamma \in \Gamma$.



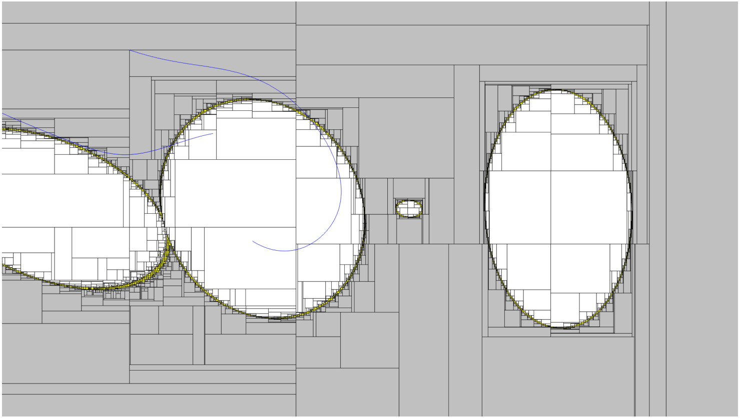


Figure: Tube Approximation

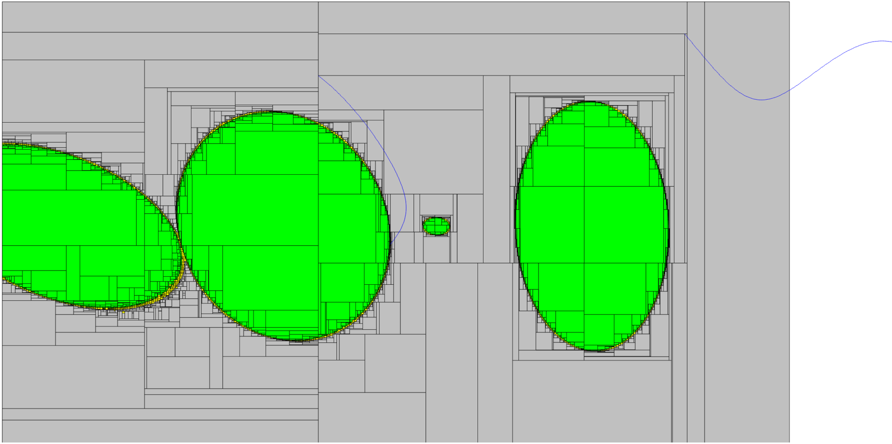


Figure: Tube approximation

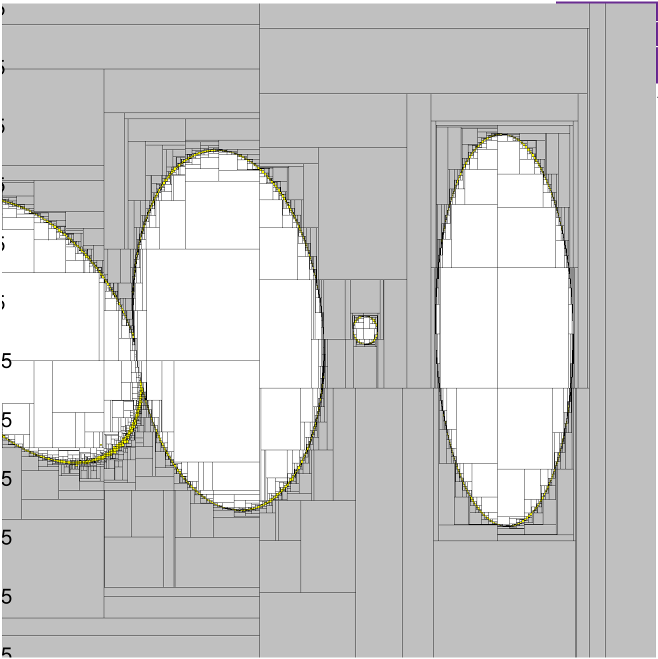


Figure: Initial approximation of the viable set

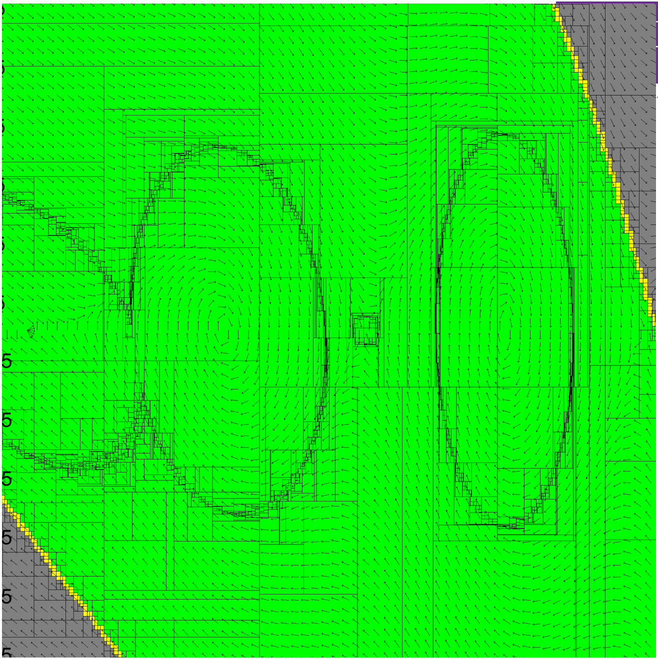


Figure: Resultant viability kernel with $[\gamma] = [0, 0]$

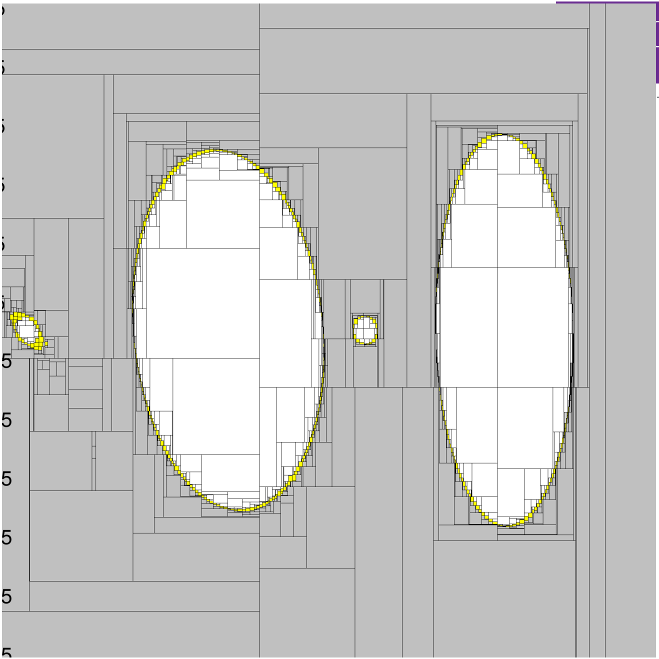


Figure: Resultant viability kernel with $[\gamma] = [-0.3, 0.3]$

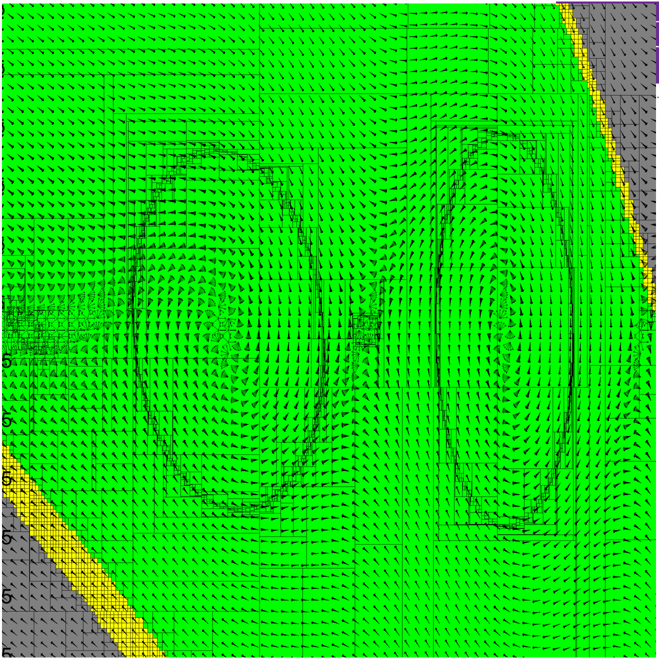


Figure: Resultant viability kernel with $[\gamma] = [-0.3, 0.3]$

Conclusions

- With this method we can check for boxes bigger in size if they are viable or not; compared with guarateed integration method where small sized boxes are used to avoid wrappig effect.
- The proposed method can be used to find a system with bouded uncertinties.

References

[1] JP. Aubin, *Viability Theory*, Birkhauser, Boston, 1991.

[2] D. Monnet, J. Ninin and L. Jaulin, Computing an inner and an outer approximation of viability kernels, *Reliable Computing* 22, 138-148, 2016.

[3] Luc Jaulin, Daniel Lopez, Vincent Le Doze, Stéphane LeMenec, Jordan Ninin, Gilles Chabert, Mohamed Saad Ibnseddik, Alexandru Stancu, Computing capture tubes, *Scientific Computing, Computer Arithmetic, and Validated Numerics, Lecture Notes in Computer Science. Springer. M. Nehmeier, J.W. von Gudenberg, W. Tucker (Eds) Vol. 9553, pages 209-224.*