Verified solution of an optimal control problem for elastic rod motions based on the Ritz method

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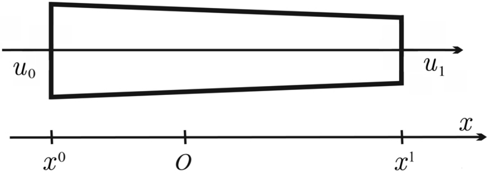


Parameters of the rod:

$s(t,x)$ – normal forces,
$p(t,x)$ – momentum density,
$w(t,x)$ – displacements,
$\kappa(x)$ – tension stiffness,
$\rho(x)$ – linear density.

Initial-boundary value problem:

$$p(t,x) = \rho(x) w_t(t,x) \quad \text{and} \quad s(t,x) = \kappa(x) w_x(t,x), \tag{1}$$

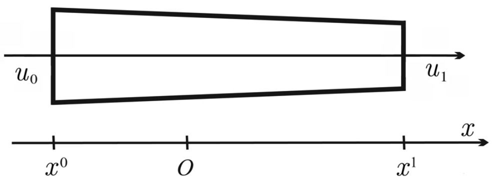
$$p_t(t,x) = s_x(t,x) + f(t,x), \tag{2}$$

$$w(0,x) = w_0(x) \quad \text{and} \quad p(0,x) = p_0(x), \tag{3}$$

$$w(t,x^0) = u_0(t) \quad \text{and} \quad s(t,x^1) = u_1(t). \tag{4}$$

Optimization in dynamics

Weighted minimization of the mean and terminal energies of the rod



Control parameters:

$W(t)$ – mechanical energy,
J_1 – mean energy,
J_2 – terminal energy,
T – terminal time instant.

Optimal control problem over a fixed time horizon:

$$J[w, p, s, u_0^*, u_1^*] = \min_{u_0, u_1 \in L_2(0,T)} J[w, p, s, u_0, u_1], \quad J = \gamma_1 J_1 + \gamma_2 J_2,$$

$$J_1 = \frac{1}{T}\int_0^T W(t)\,\mathrm{d}t, \quad J_2 = W(T), \quad W = \frac{1}{2}\int_{x^0}^{x^1} \left(\kappa w_x^2 + \rho^{-1} p^2\right) \mathrm{d}x$$

subject to the PDE constraints (1)–(4).

Scaled functions of constitutive relations:

$$g := \sqrt{\rho} w_t - \sqrt{\rho^{-1}} p \quad \text{and} \quad h := \sqrt{\kappa} w_x - \sqrt{\kappa^{-1}} s \quad \Rightarrow$$
$$g(t, x) = 0 \quad \text{and} \quad h(t, x) = 0.$$

Change of the unknown variables:

$$w(t, x) = q(t, x) + w_0(x) + u_0(t) - u_0(0), \quad u_0(0) = w_0(0)$$
$$p(t, x) = r_x(t, x) + p_0(x) + \int_0^t f(\tau, x)\, \mathrm{d}\tau,$$
$$s(t, x) = r_t(t, x) + u_1(t).$$

$q(t, x)$ – relative displacements
$r(t, x)$ – dynamic potential.

Minimization of constitutive functional:

$$J_3\big[q^*, r^*, u_0, u_1\big] = \min_{q,r\in H^1(\Omega)} J_3\big[q, r, u_0, u_1\big] = 0,$$

$$J_3 = \frac{1}{T}\int_\Omega \varphi(t,x)\,\mathrm{d}\Omega = 0 \quad \text{with} \quad \varphi = \frac{g^2}{2} + \frac{h^2}{2},$$

subject to the homogeneous constraints

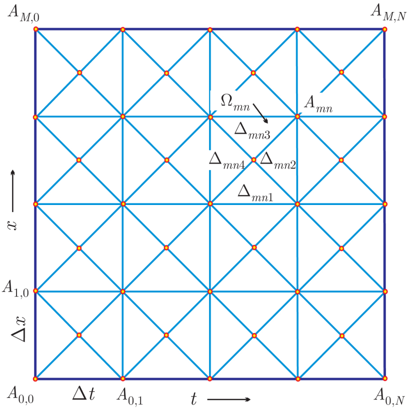
$$q(0,x) = 0 \quad \text{and} \quad r(0,x) = 0,$$
$$q(t,x^0) = 0 \quad \text{and} \quad r(t,x^1) = 0.$$

$\varphi(t,x)$ – local error distribution

$\Omega = (0,T) \times (x^0, x^1)$ – time-space domain

Finite element method

Regular triangulation of the space-time domain Ω



$$t_i = \frac{iT}{N}, \; x_i = x^0 + \frac{ix^1}{M}$$

Piecewise polynomial splines:

$$q = \sum_{k+l \le K} q_{mnj}^{(kl)} t^k x^l, \; (t,x) \in \Delta_{mnj},$$

$$r = \sum_{k+l \le K} r_{mnj}^{(kl)} t^k x^l, \; (t,x) \in \Delta_{mnj}.$$

Finite dimensional representation:

$$q(t,x) = \mathbf{z}^{\mathrm{T}} \mathbf{q}(t,x) \in C_0^0(\Omega),$$

$$r(t,x) = \mathbf{z}^{\mathrm{T}} \mathbf{r}(t,x) \in C_0^0(\Omega),$$

$$\mathbf{z} \in \mathbb{R}^{N_g}, \quad N_g = 4MNK^2.$$

Isoperimetric tolerance condition:

$$J_3[\mathbf{z}, u_0, u_1] = \varepsilon \ll J_2[\mathbf{z}, u_0, u_1] \ll J_1[\mathbf{z}, u_0, u_1].$$

Equivalent optimal control problem:

$$\tilde{J}[\mathbf{z}^*, u_0^*, u_1^*] = \min_{u_0, u_1 \in L_2(0,T)} \tilde{J}[\tilde{\mathbf{z}}(u_0, u_1), u_0, u_1],$$

$$\tilde{J} = \gamma_1 J_1 + \gamma_2 J_2 + \gamma_3 J_3,$$

$$\tilde{\mathbf{z}}(u_0, u_1)] = \arg\min_{\mathbf{z}} J_3\left[\mathbf{z}, u_0, u_1\right] \geq 0.$$

Approximation error:

$$\Delta = J_3[\mathbf{z}^*, u_0^*, u_1^*] J_1^{-1}[\mathbf{z}^*, u_0^*, u_1^*], \quad \mathbf{z}^* = \tilde{\mathbf{z}}(u_0^*, u_1^*)$$

Control discretization:

$u_i = \sum_{k=1}^{K} v_{j(k)} t^{k-1}$, where $j = (1-i)KN + (n-1)K + k$,

for $t \in (t_n, t_{n+1})$ with $t_n = nT/N$ and $n = 1, \ldots, N$.

$\mathbf{u} = [v_1 \; \ldots \; v_J]^{\mathrm{T}}$ is the control vector with $J = 2KN$.

Constitutive functional:

$$\tilde{J}_3(\mathbf{z}, \mathbf{u}) = \frac{1}{2}\mathbf{y}^{\mathrm{T}}\mathbf{F}\mathbf{y} + \mathbf{f}^{\mathrm{T}}\mathbf{y} + f \to \min_{\mathbf{z}},$$

$$\text{where} \quad \mathbf{F} = \begin{bmatrix} \mathbf{F}_{zz} & \mathbf{F}_{zu} \\ \mathbf{F}_{zu}^{\mathrm{T}} & \mathbf{F}_{uu} \end{bmatrix}, \quad \mathbf{f} = \begin{bmatrix} \mathbf{f}_z \\ \mathbf{f}_u \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} \mathbf{z} \\ \mathbf{u} \end{bmatrix} \quad \Rightarrow$$

$$\tilde{\mathbf{z}} = -\mathbf{F}_{zz}^{-1}\left(\mathbf{F}_{zu}\mathbf{u} + \mathbf{f}_z\right).$$

Numerical algorithm of control optimization

Finding the optimal control parameters

Cost functional:

$$\tilde{J}(\mathbf{u}) = \tilde{J}\left(\tilde{\mathbf{z}}(\mathbf{u}), \mathbf{u}\right) = \frac{1}{2}\mathbf{u}^{\mathrm{T}}\mathbf{V}\mathbf{u} + \mathbf{v}^{\mathrm{T}}\mathbf{u} + v \to \min_{\mathbf{u}}$$

Optimal control parameters:

$\mathbf{u}^* = -\mathbf{V}^{-1}\mathbf{v}.$

Optimized control law:

$\mathbf{z}^* = \tilde{\mathbf{z}}(\mathbf{u}^*),\ u_i^* = u_i(t, \mathbf{z}^*, \mathbf{u}^*),$
$w^* = w(t, x, \mathbf{z}^*, \mathbf{u}^*),\ p^* = p(t, x, \mathbf{z}^*, \mathbf{u}^*),\ s^* = (t, x, \mathbf{z}^*, \mathbf{u}^*).$

Structural and control parameters:

Structure characteristics: $\rho = \kappa = x^1 = 1,\ x^0 = 0,\ f(t,x) \equiv 0.$
Control characteristics: $T \geq 2,\ \gamma_1 \neq 0.$
Control inputs: $u_0(t) \equiv 0,\ u_1(t) = u(t).$

Optimal control law:

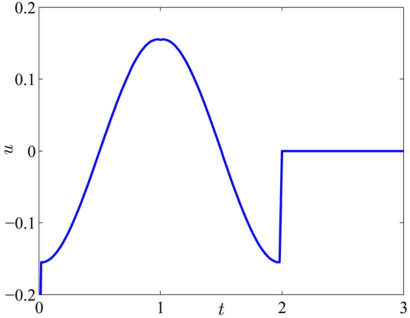
$$u^*(t) = \begin{cases} \bar{u}^*(t) & \text{for } \ t \in [0,2] \\ 0 & \text{for } \ t > 2 \end{cases}$$

$$2\bar{u}^*(t) = \begin{cases} r_0'(t-1) - w_0'(t-1) & \text{for } \ t \in [0,1] \\ r_0'(t-1) + w_0'(t-1) & \text{for } \ t \in (1,2] \end{cases} .$$

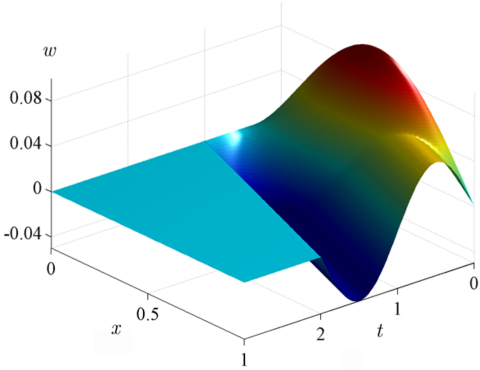
Here, $r_0(t) = -\int_t^1 p_0(\tau)\,\mathrm{d}\tau.$

Optimal control for the uniform rod

Displacement and control functions



Optimal input (force).



Optimal displacements.

Structural and control parameters:

Rod characteristics: $\rho = x^1 = 1$, $x^0 = 0$, $f(t,x) \equiv 0$,

$$\kappa(x) = \begin{cases} 2.25 & \text{for } x < 0.5 \\ 0.5625 & \text{for } x > 0.5 \end{cases}.$$

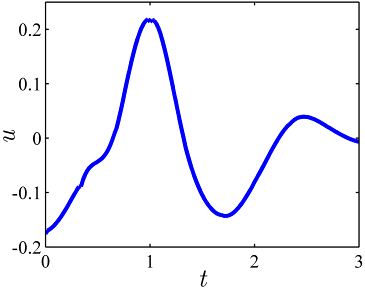
Initial conditions: $p_0(x) \equiv 0$, $w_0(x) = \sin(\pi x)$.

Control characteristics: $T = 3$, $\gamma_1 = \gamma_2 = 1$, $\gamma_3 = 10^6$.
Control inputs: $u_0(t) \equiv 0$, $u_1(t) = u(t)$.

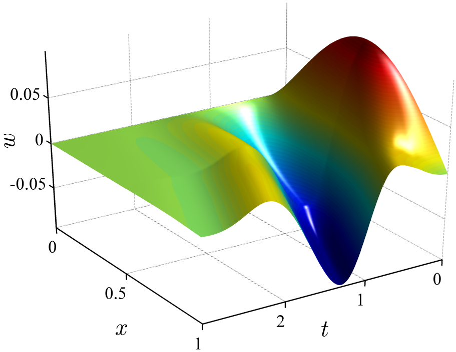
Mesh sizes: $N = 9$ in time and $M = 2$ in space.
Polynomial degree: $K = 5$.

Optimal control for a nonuniform rod

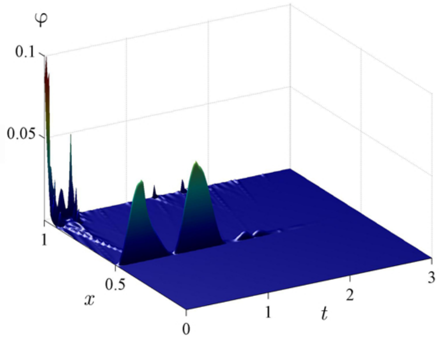
Displacement and control functions



Optimal input (force).



Optimal displacements.



Error distribution

Conventional optimization:

Weights: $\gamma_1 = \gamma_2 = 1,\ \gamma_3 = 0$
Mean energy: $J_1 = 0.255$
Terminal energy: $J_2 = 5.9 \cdot 10^{-6}$
Error energy: $J_3 = 8.58 \cdot 10^{-5}$
Relative error: $\Delta = 7.2 \cdot 10^{-4}$

Regularized optimization:

Weights: $\gamma_1 = \gamma_2 = 1,\ \gamma_3 = 10^6$
Mean energy: $J_1 = 0.287.$
Terminal energy: $J_2 = 2.19 \cdot 10^{-5}$
Error energy: $J_3 = 4.88 \cdot 10^{-7}$
Relative error: $\Delta = 1.7 \cdot 10^{-6}$
$\Rightarrow \quad J_3 \ll J_2 \ll J_1$

Conclusions

- A control strategy for energy optimization in structural dynamics has been proposed.
- Motions of a nonuniform elastic rod are considered as an example.
- The optimization algorithm is based on the method of integro-differential relations, which combines a variational formulation of dynamic problems with the finite element method.
- The verification of optimal control laws has been performed taking into account the explicit local and integral error estimates.