Transformation of Dynamic Systems Into a Cooperative Form to Exploit Advantages in Interval-based Controller Design

SWIM 2018: Summer Workshop on Interval Methods
Rostock, Germany, July 27th, 2018

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Why cooperativity?

To simplify

- computation of guaranteed state enclosures
- design of interval observers
- forecasting worst-case bounds for selected system outputs in predictive control
- identification of unknown parameters
- ...

Avoiding the use of general-purpose, set-valued solvers

Overestimation due to the wrapping effect may lead to (interval) bounds that are much wider than the actually reachable sets of states.
Consider the autonomous system

\[ \dot{x}(t) = f(x(t)) , \ x \in \mathbb{R}^n \]

**Criterion for cooperativity**

Jacobian matrix

\[ J = \frac{\partial f(x)}{\partial x} \]

with all off-diagonal elements \( J_{i,j} \), \( i, j \in \{1, \ldots, n\} \), \( i \neq j \) strictly non-negative according to

\[ J_{i,j} \geq 0 , \ i, j \in \{1, \ldots, n\} , \ i \neq j \]
Consider the autonomous system

\[ \dot{x}(t) = f(x(t)), \quad x \in \mathbb{R}^n \]

**Positivity of the system**

Guarantee that state trajectories \( x(t) \) starting in the positive orthant

\[ \mathbb{R}_+^n = \{ x \in \mathbb{R}^n | x_i \geq 0 \quad \forall i \in \{1, \ldots, n\} \} \]

stay in this positive orthant for all \( t \geq 0 \) because

\[ \dot{x}_i(t) = f_i(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n) \geq 0 \]

holds for all components \( i \in \{1, \ldots, n\} \) of the state vector as soon as the state \( x_i \) reaches the value \( x_i = 0 \)
Interval representation of domain of reachable states

\[
\begin{bmatrix}
[x_1] = [x](t) = \\
\vdots \\
[x_n] = [x](t) = \\
\end{bmatrix}
\begin{bmatrix}
[x_1(t); \overline{x}_1(t)] \\
\vdots \\
[x_n(t); \overline{x}_n(t)]
\end{bmatrix}
\]

with the initial states

\[
[x_0] = [x](0) = \\
\vdots \\
\begin{bmatrix}
[x_1(0); \overline{x}_1(0)] \\
\vdots \\
[x_n(0); \overline{x}_n(0)]
\end{bmatrix}
\]

and the vector components \([x_i] = [x_i; \overline{x}_i], i \in \{1, \ldots, n\}\), where \(\inf ([x_i]) = x_i\) is the infimum
\(\sup ([x_i]) = \overline{x}_i\) is the supremum
Cooperative System Models Derived From First-Principle

Derivation of the ODEs

\[ \dot{x}_i = - \sum_{j=1}^{n} p_{ij} x_i + \sum_{j=1, i \neq j}^{n} p_{ji} x_j \]

with \( p_{ii} \in \mathbb{R}, p_{ij} \geq 0 \) and \( p_{ji} \geq 0, i \neq j \)
Cooperative System Models Derived From First-Principle

Figure: Graphical representation of a dynamic system.

State-space representation

\[
\dot{x} = \begin{bmatrix}
-\sum_{j=1}^{n} p_{1j} & p_{21} & \cdots & p_{n1} \\
p_{12} & -\sum_{j=1}^{n} p_{2j} & \cdots & p_{n2} \\
\vdots & \vdots & \ddots & \vdots \\
p_{1n} & p_{2n} & \cdots & -\sum_{j=1}^{n} p_{nj}
\end{bmatrix} x,
\]
Reformulation into a quasi-linear state-space representation

\[ \dot{x} = f(x, u) \]

For linear systems the state equations are equivalent to

\[ \dot{x} = A \cdot x + B \cdot u \]

and for non-linear formulations a quasi-linear form (by factoring out selected state variables) is obtained

\[ \dot{x} = A(x) \cdot x + B(x) \cdot u \]

Cooperativity

Here, \( A \) or \( A(x) \) is supposed to be Metzler and Hurwitz for asymptotically stable systems
Preparations for the transformation

\[ x = x_s = 0 \quad \text{desired operating state} \]
\[ u = u_s = 0 \quad \text{without loss of generality for the steady-state input} \]

with the feedback controller according to

\[ u = -Kx \quad \text{or} \quad u = -K(x) \cdot x \]

leading to

\[ \dot{x} = (A(x) - B(x) \cdot K(x)) \cdot x = A_C(x) \cdot x \]
General transformation

\[ z(t) = \Theta^{-1}x(t) \quad \text{with} \quad \dot{z}(t) = N \cdot z(t) \]

For general applications without diagonally dominant system matrices, the transformation consists of

\[ \tilde{z}(t) = \tilde{T}^{-1}x(t) \]

to get a diagonally dominant system matrix and

\[ z(t) = T^{-1}\tilde{z}(t) = T^{-1} \cdot \tilde{T}^{-1}x(t) = (\tilde{T} \cdot T)^{-1}x(t) \]

to ensure a Metzler structure, resulting in the overall transformation matrix \( \Theta = \tilde{T} \cdot T \)
Structure of the transformation matrix

\[ \Theta \] may be a **time-invariant** or **time-varying** matrix according to the following distinction.
Systems With Purely Real Eigenvalues

Preliminary

\[ Z_a - \Delta \leq Z := A_C \leq Z_a + \Delta \]

with \( \Delta \), which consists of the (symmetric) worst-case bounds of all entries in \( [A]_C \) and \( Z_a = Z_a^T \) as a symmetric midpoint matrix and

\[ R = \mu E_n - \Gamma \]

as Metzler matrix, which has the same eigenvalues as \( Z_a \)

with

- \( \mu \in \mathbb{R} \) constant
- \( \Gamma \in \mathbb{R}^{n \times n} \) diagonal matrix
- \( E_n \in \mathbb{R}^{n \times n} \) matrix with all elements equal to 1 and
- \( \Gamma = \rho I_n \) with \( \rho > \mu \) and the identity matrix \( I \) of order \( n \)
Assumption

If

$$\text{eig}(R) = \text{eig}(Z_a)$$

there exists an orthogonal matrix $$S \in \mathbb{R}^{n \times n}$$ such that $$S^T Z S$$ or $$\Theta^T Z \Theta$$, respectively, is Metzler provided that

$$\mu > n||\Delta||_{\max},$$

where $$||\Delta||_{\max}$$ denotes the maximum absolute value of $$\Delta$$. 

J. Kersten et al.: Transformation of Dynamic Systems Into a Cooperative Form
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\[ \mu > n ||\Delta||_{\text{max}} , \]

where \( ||\Delta||_{\text{max}} \) denotes the maximum absolute value of \( \Delta \)

Aim

Computationally feasible optimization problem formulated with LMI constraints to find a suitable transformation matrix \( S \)
Choosing a diagonal matrix $Z_a$

If the system is

1. diagonally dominant: $Z_a$ represents the diagonal entries of the original system matrix

2. not diagonally dominant: $Z_a$ is a diagonal matrix with the asymptotically stable, real eigenvalues of $\text{mid}\{[A]_C\}$, which is determined by a new matrix

$$\tilde{A}_C = \tilde{T}^{-1} A_C \tilde{T}$$

such that the element-wise defined interval midpoint matrix $\text{mid}\{[A]_C\}$ is transformed into a diagonal structure (except for numerical round-off errors). If $\text{mid}\{[A]_C\}$ possesses $n$ linearly independent real-valued eigenvectors, their floating-point approximation is used to define the matrix $\tilde{T}$. 
Choosing $\Delta$

$$\delta = \max (\|[A]_C - Z_a\|) \quad \text{or} \quad \delta = \max (\|\hat{A} - Z_a\|)$$

with $\Delta = \delta \cdot E_n$

Further specifications

$$\mu^* = n \|\Delta\|_{\text{max}}$$

marks the lower bound for $\mu$ and

$$R = S^T Z_a S \quad \text{and} \quad S^T S = I$$

need to be satisfied
Reformulation Into an Optimization Problem

Orthogonality of $S$

\[-R + S^T Z_a S \succ 0 \quad \text{and} \quad I - S^T S \succ 0\]

is converted by application of the Schur complement formula according to

\[
\begin{bmatrix}
-R & S^T \\
S & -Z_a^{-1}
\end{bmatrix} \succ 0 \quad \text{and} \quad \begin{bmatrix}
I & S^T \\
S & I
\end{bmatrix} \succ 0
\]
Reformulation Into an Optimization Problem

Known specifications to other variables

\[
\mathbf{R} = \bar{\mu} \mathbf{E}_n - \Gamma, \quad \bar{\mu} > \mu
\]

where the LMI constraints

\[
\Gamma > 0 \quad \text{and} \quad \mathbf{R}^T \mathbf{Q} + \mathbf{Q} \mathbf{R} < 0
\]

with \( \mathbf{Q} > 0 \) (Hurwitz stability of \( \mathbf{R} \))

Overall cost function

\[
J = \text{tr}(\Gamma) + \text{tr}(\mathbf{Z}_a \mathbf{S} - \tilde{\mathbf{S}} \mathbf{R}) - \kappa \cdot \text{tr}(\tilde{\mathbf{S}}^T \mathbf{S} - \mathbf{I})
\]

with the problem-dependent parameter \( \kappa > 0 \) and the solution of the last successful evaluation of the LMI-constrained optimization task \( \tilde{\mathbf{S}} \)
Systems With Purely Conjugate-Complex Eigenvalues

Preliminaries

- Generally only time-varying transformations possible (exception are exactly known systems)
- Uncertainty is mapped into the position of the eigenvalues
Systems With Purely Conjugate-Complex Eigenvalues

\[ \lambda_1 \neq \lambda_1^* \]

\[ \omega_1 \neq -\omega_1 \]

**Figure:** Possible positions of conjugate-complex eigenvalues.

**Interval hull**

\[ [\sigma_i] = [\sigma_i; \bar{\sigma}_i] \text{ and } [\omega_i] = [\omega_i; \bar{\omega}_i] \]
The presented approach is only valid for disjoint eigenvalue domains.

**Transformation matrix**

The number of considered eigenvalues is reduced to

$$\tilde{n} = \frac{n}{2}$$

for a system with \(n\) states, because of conjugate-complex pairs

$$\tilde{T} = \begin{bmatrix} \tilde{T}_1, \ldots, \tilde{T}_{\tilde{n}} \end{bmatrix}, \text{ where } \tilde{T}_j = [\Re\{[v_j]\}, \Im\{[v_j]\}]$$

with \(j \in \{1, \ldots, \tilde{n}\}\)
Transformed system

Results in the real-valued Jordan canonical form

$$\tilde{A} = \text{blkdiag} \left( \tilde{A}_1, \ldots, \tilde{A}_{\tilde{n}} \right) \text{ with } \tilde{A}_j = \begin{bmatrix} \sigma_j & \omega_j \\ -\omega_j & \sigma_j \end{bmatrix}.$$  

The time-variant transformation is done by

$$z = T^{-1}(t) \cdot \tilde{z} \text{ with }$$

$$T^{-1}(t) = \text{blkdiag} \left( T_1^{-1}(t), \ldots, T_{\tilde{n}}^{-1}(t) \right) = T^T(t)$$

and the orthogonal blocks

$$T_j = \begin{bmatrix} \cos(\omega_j t) & \sin(\omega_j t) \\ -\sin(\omega_j t) & \cos(\omega_j t) \end{bmatrix}$$

for $j \in \{1, \ldots, \tilde{n}\}$
Resulting state-space representation of the system

\[
\dot{\mathbf{z}} = \mathbf{T}^T(t) \cdot \tilde{\mathbf{z}} + \mathbf{T}^T(t) \cdot \dot{\tilde{\mathbf{z}}}
\]

\[
= \left[ \left( \frac{d\mathbf{T}^T(t)}{dt} + \mathbf{T}^T(t) \tilde{\mathbf{A}} \right) \mathbf{T}(t) \right] \mathbf{z} = \mathbf{N} \cdot \mathbf{z}
\]

Regarding the structure of \( \mathbf{N} \)

Symbolic simplifications in terms of the exact values \( \omega_j^* \)

\[
\mathbf{N} = \text{blkdiag} \left( \sigma_1 \mathbf{I}, \ldots, \sigma_{\tilde{n}} \mathbf{I} \right), \quad \mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]
Resulting state-space representation of the system

\[
\dot{\tilde{z}} = T^T(t) \cdot \tilde{z} + \dot{T}^T(t) \cdot \tilde{\dot{z}} \\
= \left[ \frac{d T^T(t)}{dt} + T^T(t) \tilde{A} \right] T(t) \right] z = N \cdot z
\]

Asymptotic stability

Since \( N \) depends on the system’s eigenvalues, Hurwitz stability is guaranteed for \( \sigma_j < 0 \). Extrema of the conjugate-complex eigenvalues are obtained by building the hull over their real and imaginary parts

\[
[\sigma_j] = [\min(\sigma_j); \max(\sigma_j)] , \\
[\omega_j] = [\min(\omega_j); \max(\omega_j)] .
\]
System With Purely Real Eigenvalues

![Diagram of an inverted pendulum on a moving carriage](image)

**Figure:** Control of an inverted pendulum on a moving carriage.

### Nonlinear differential equations

\[
\begin{align*}
    ma^2 \cdot \ddot{\alpha} - ma \cdot \cos(\alpha) \cdot \ddot{x} - mga \cdot \sin(\alpha) &= 0, \\
    (M + m) \cdot \ddot{x} - ma \cdot \cos(\alpha) \cdot \dddot{\alpha} + ma \cdot \sin(\alpha) \cdot \dot{\alpha} &= F
\end{align*}
\]
Quasi-linear state-space representation

Introducing an underlying velocity control with \( u \) as the desired carriage velocity

\[
T_1 \cdot \ddot{x} + \dot{x} = u
\]

\[
\dot{x} = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
g \cdot \sin(\alpha) & 0 & 0 & -\frac{\cos(\alpha)}{T_1 a} \\
0 & 0 & 0 & -\frac{1}{T_1}
\end{bmatrix} x + \begin{bmatrix}
0 \\
0 \\
\frac{\cos(\alpha)}{T_1 a} \\
\frac{1}{T_1}
\end{bmatrix} u
\]

\[
y = [-a \cdot \sin(\alpha) \ 1 \ 0 \ 0] x , \quad \sin(\alpha) = \frac{\sin(\alpha)}{\alpha}
\]

with the state vector \( x = [\alpha \ x \ \dot{\alpha} \ \dot{x}]^T \) and the system input \( u \).
Controlled system for $[\alpha] = [51.4^\circ; 51.7^\circ]$

$$A_C(\alpha) = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-\alpha_{31}(\alpha) - k_1\beta_3(\alpha) & -k_2\beta_3(\alpha) & -k_3\beta_3(\alpha) & \alpha_{34}(\alpha) - k_4\beta_3(\alpha) \\
-k_1\beta_4 & -k_2\beta_4 & -k_3\beta_4 & \alpha_{44} - k_4\beta_4
\end{bmatrix}$$

$$[a]_{31} = [-1.25; -1.23] \cdot 10^3$$

$$[a]_{33} = [-2.55; -2.52] \cdot 10^2$$

$$[a]_{41} = [-4.13; -4.12] \cdot 10^2$$

$$[a]_{43} = [-81.49; -81.48]$$

$$[a]_{32} = [12.81; 12.91]$$

$$[a]_{34} = [1.07; 1.09] \cdot 10^2$$

$$[a]_{42} = [4.13; 4.14]$$

$$[a]_{44} = [34.70; 34.71]$$
**Transformation matrix**

\[ \Theta = VS = \begin{bmatrix} -0.016 & -0.170 & -0.016 & 0.155 \\ 0.998 & 0.694 & -0.104 & -0.462 \\ 0.056 & 0.309 & 1.095 & -0.012 \\ 0.283 & -0.253 & -0.504 & 0.740 \end{bmatrix} \]
Transosed system

\[ \mathbf{N} = \tilde{\mathbf{A}}_C \in 1 \cdot 10^2 \begin{bmatrix}
[a]_{11} & [a]_{12} & [a]_{13} & [a]_{14} \\
[a]_{21} & [a]_{22} & [a]_{23} & [a]_{24} \\
[a]_{31} & [a]_{32} & [a]_{33} & [a]_{34} \\
[a]_{41} & [a]_{42} & [a]_{43} & [a]_{44}
\end{bmatrix} \]

with

\[
[a]_{11} = [-2.124; -2.100] \\
[a]_{21} = [0.152; 0.166] \\
[a]_{31} = [0.064; 0.067] \\
[a]_{41} = [0.026; 0.038] \\
[a]_{12} = [0.143; 0.175] \\
[a]_{22} = [-0.057; -0.038] \\
[a]_{32} = [0.001; 0.005] \\
[a]_{42} = [0.001; 0.018] \\
[a]_{13} = [0.061; 0.070] \\
[a]_{23} = [0.000; 0.006] \\
[a]_{33} = [-0.010; -0.008] \\
[a]_{43} = [0.000; 0.005] \\
[a]_{14} = [0.016; 0.047] \\
[a]_{24} = [0.000; 0.018] \\
[a]_{34} = [0.000; 0.004] \\
[a]_{44} = [-0.024; -0.008]
\]
System With Purely Conjugate-Complex Eigenvalues

Figure: Mechanical model of the wind turbine with an elastic tower.
Ordinary differential equations

\[
\begin{bmatrix}
\dot{x} \\
\ddot{x}
\end{bmatrix} =
\begin{bmatrix}
0 & I \\
-M_{TB}^{-1}K_{TB} & -M_{TB}^{-1}(D_{TB} + h_{TB}k_{T}^T) \\
\end{bmatrix}
\begin{bmatrix}
x \\
\dot{x}
\end{bmatrix}
\]

with uncertainties in \(M_{TB}, D_{TB}\) and \(K_{TB}\) due to their dependency on \(\omega_R, k_{dT}\) and \(k_{dB}\)

Parameter domains

\[
[\omega_R] = [0.7; 1.4] \, s^{-1}
\]
\[
[k_{dT}] = [2.5; 3.5] \cdot 10^{-2} \, N \cdot s/m
\]
\[
[k_{dB}] = [0.5; 1.5] \cdot 10^{-2} \, N \cdot s/m
\]
Transformation

\[ [\mathbf{N}] = \text{diag} ([\sigma_1], [\sigma_1], [\sigma_3], [\sigma_3]) \]

with

\[ [\sigma_1] = [-0.105; -0.016] \]
\[ [\omega_1] = [3.893; 5.310] \]
\[ [\sigma_3] = [-0.068; -0.040] \]
\[ [\omega_3] = [1.875; 1.908] \]
Transformation of the initial conditions

\[ \tilde{T} = [v_1^R, v_1^S, v_3^R, v_3^S] \]

\[
[x](0) = \begin{bmatrix}
1.25; 1.5 \\
0.25; 0.5 \\
0 \\
0
\end{bmatrix} \quad \xrightarrow{\tilde{T}} \quad [z](0) = \begin{bmatrix}
-0.761; 0.335 \\
-9.792; 1.608 \\
-0.297; 0.619 \\
2.609; 5.474
\end{bmatrix}
\]
System with real and complex eigenvalues

Figure: Mechanical model of the stacker crane.
Ordinary differential equations

\[ M\ddot{q} + D\dot{q} + Kq = h \cdot [F_{SM} - F_{SR}(\dot{y}_S)] \]

with uncertainties in \( M \) and \( K \) due to their dependency on \( \kappa \), a dimensionless Parameter to consider the varying vertical position of the payload \( x_K \)

\[ \kappa = \frac{x_K}{l} \]

Parameter domains

\[ [\kappa] = [0.45; 0.54] \text{ m} \]
**Transformation**

\[
[N] = \text{diag} ([\sigma_1], [\sigma_2], [\sigma_2], [\sigma_4], [\sigma_5], [\sigma_5])
\]

with

\[
[\sigma_1] = [-601.4; -560.1]
\]
\[
[\sigma_2] = [-27.4; -25.7]
\]
\[
[\omega_2] = [125.9; 127.5]
\]
\[
[\sigma_4] = [-7.5; -6.8]
\]
\[
[\sigma_5] = [-20.2; -17.5]
\]
\[
[\omega_5] = [19.8; 22.5]
\]
Conclusions

- Using advantages of cooperative systems
  - avoiding the wrapping effect
  - simulating lower and upper bounds individually
  - reflecting the characteristics of exactly known systems
- purely real eigenvalues: LMI approach to satisfy given requirements
- complex eigenvalues: symbolically proven solution
Conclusions

- Using advantages of cooperative systems
  - avoiding the wrapping effect
  - simulating lower and upper bounds individually
  - reflecting the characteristics of exactly known systems
- purely real eigenvalues: LMI approach to satisfy given requirements
- complex eigenvalues: symbolically proven solution

Outlook

- Optimization of the line-search procedure for the parameter $\mu$
- Performance improvement for higher-dimensional applications
- Extensions to systems with real and conjugate-complex eigenvalues as well as multiple eigenvalues in a joint approach combining both presented procedures
Thank you for your attention!