Guaranteed Bounding Zones for GNSS Positioning by Geometrical Constraints

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DFG Research Training Group (GRK2159)
i.c.sens – Integrity and Collaboration in dynamic sensor networks
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Motivation

- Geometrical constrains provide rigorous and reliable computing
  - Smallest guaranteed bounding zones
  - Fault detection and exclusion
  - Minimum detectable bias
  - Inconsistency measures
Least Squares Adjustment

\[ \rho = \sqrt{(x_{sv} - x_u)^2 + (y_{sv} - y_u)^2 + (z_{sv} - z_u)^2 + \delta t + w = f(x)} \]

\[ \rho_1 = \sqrt{(x_{sv1} - x_u)^2 + (y_{sv1} - y_u)^2 + (z_{sv1} - z_u)^2 + \delta t} \]
\[ \rho_2 = \sqrt{(x_{sv2} - x_u)^2 + (y_{sv2} - y_u)^2 + (z_{sv2} - z_u)^2 + \delta t} \]
\[ \rho_3 = \sqrt{(x_{sv3} - x_u)^2 + (y_{sv3} - y_u)^2 + (z_{sv3} - z_u)^2 + \delta t} \]
\[ \rho_4 = \sqrt{(x_{sv4} - x_u)^2 + (y_{sv4} - y_u)^2 + (z_{sv4} - z_u)^2 + \delta t} \]

\[ d\rho = OMC = f(x_0) - \rho = \hat{\rho} - \rho \]
\[ d\hat{x} = (A^T P A)^{-1} A^T P \cdot d\rho \]

where \( A = \left[ \frac{\partial f}{\partial \chi} \right]_{x_0} \)

\[ \hat{x} = x_0 + d\hat{x} \]
Primal-Dual Polytope

- Hyperplane is a set of the form \( \{x \mid ax = b\} \)

- \( H - \text{Polytope} = \{x \mid a_i x \leq b_i, i = 1, \ldots, n, c_i x = d_i, i = 1, \ldots, p\} \)

- \( V - \text{Polytope} = \text{conv}(X) = \{\sum_{i=1}^{n} \lambda_i x_i | \lambda_i \geq 0, \sum_{i=1}^{n} \lambda_i = 1\} \)

- \([d\rho] = [d\rho - \Delta, d\rho + \Delta] \)
Primal-Dual Polytope

- \( d\rho - \Delta \leq Ad\hat{x} \leq d\rho + \Delta \)

- \[
\begin{align*}
    Ad\hat{x} & \leq d\rho + \Delta \\
    -Ad\hat{x} & \leq -d\rho + \Delta
\end{align*}
\] \( \leftrightarrow \) \( Bd\hat{x} \leq b \)

- where, \( B = \begin{bmatrix} A \\ -A \end{bmatrix} \), \( b = \begin{bmatrix} d\rho + \Delta \\ -d\rho + \Delta \end{bmatrix} \)
Derivation of Observation Interval Error Bounds

- Three different ways to set the error bounds
  - Probabilistic approach with prior integrity risk
  - Sensitivity analysis of the measurement correction
  - Expert knowledge

- Error bounds and navigation geometry define the volume and the shape of the bounding zone i.e. the polytope
Impact of Geometry

Error Bound = 4 m, Area = 64 m²

Error Bound = 4 m, Area = 90.5097 m²

Error Bound = 4 m, Area = 579.5446 m²
Impact of Random Noise

Black : Error Free; Colored: AWGD ($\sigma = 2$ m, $\mu = 0$)

<table>
<thead>
<tr>
<th>$\Delta$ = 16 m</th>
<th>$\Delta$ = 12 m</th>
<th>$\Delta$ = 8 m</th>
<th>$\Delta$ = 4 m</th>
<th>Error Free</th>
<th>True Pos</th>
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Real Data: Error Indicating Polytope

$\Delta = 6m$
Point Positioning Error Analysis

Bad Geometry ➔ high error in PDP

Bias ➔ Empty set from PDP
Point Positioning Error Analysis

![Graph showing cumulative frequency of 3D coordinates error][1]

- **LSA**
- **LSA Outlier Exclusion**
- **PDP 0 relax**
- **PDP 1 relax**

[1]: https://example.com/graph.png
Impact of Biased Measurement

Tx1
Tx2
Tx1
Tx2
Tx1
Tx2
Tx3
Tx3
Tx3

No Bias
Detectable Bias
Non-Detectable Bias
Impact of Biased Measurements

- Figure a: Polytope (blue) and LOS (red) for different bias levels.
- Figure b: Polytope (blue) and LOS (red) for different bias levels.
- Figure c: Polytope (blue) and LOS (red) for different bias levels.
- Figure d: Polytope (blue) and LOS (red) for different bias levels.
Minimum Detectable Bias

\[
MDB_{Z_i} = w_{hl_Z} + \Delta_i \\
MDB_{P_i} = w_{hl_P} + \Delta_i \\
MDB_{TS,i} = \sqrt{\frac{\lambda_0}{c_i^T Q^{-1}_y (I_m - P_A) c_i}}
\]
Minimum Detectable Bias

GPS Code Internal Reliability

DOBp [m]

DOBj [m]

Epoch Index [s]
Minimum Detectable Bias

No solution from Test Statistics

No solution from Polytope
Inconsistency measures

\[ Vol_r = \frac{Vol_Z - Vol_P}{Vol_Z} \leq T_r \]

Real Data

Simulated Data AWGN(3m,0)
Inconsistency measures

Simulated Data with Different AWGN

Simulated Data with Same AWGN

- $\Delta = 6 \text{ m}, \sigma = 1 \text{ m}$
- $\Delta = 12 \text{ m}, \sigma = 2 \text{ m}$
- $\Delta = 18 \text{ m}, \sigma = 3 \text{ m}$

- $\Delta = 9 \text{ m}, \sigma = 3 \text{ m}$
- $\Delta = 18 \text{ m}, \sigma = 6 \text{ m}$
- $\Delta = 27 \text{ m}, \sigma = 9 \text{ m}$
Inconsistency measures

$\Delta = \text{6 [Meter]}$

$\Delta = \text{12 [Meter]}$

V_0 - V/N_0$

Volume $[\text{m}^3 / \text{1000}]$

Epoch Index [Second]
Conclusions

- PDP shows higher precision and accuracy than LSA
- PDP is more sensitive to the positioning geometry than LSA
- New methods to derive MDB with better performance than traditional hypothesis test statistics
  - Shape and volume of the polytope are strictly related to the geometry and noise distribution
  - PDP gives empty sets in the presence of detectable bias
  - PDP provides a inconsistency check bounding zone