An Interval Median Algebra

Stavros P. Adam[1,3], Gerasimos C. Meletiou[2,3]
and Michael N. Vrahatis[3]

[1]Dept. of Computer Engineering,
Technological Educational Institute of Epirus, Greece,
[2]Dept. of Agricultural Technology,
Technological Educational Institute of Epirus, Greece
[3] Computational Intelligence Laboratory,
Dept. of Mathematics, University of Patras, Greece

SWIM 2018 - University of Rostock

July 25, 2018

A new algebraic structure on intervals - Why?

The algebraic structure of intervals matters

A new algebraic structure on intervals - Why?

The algebraic structure of intervals matters

T. Sunaga, "Theory of an Interval Algebra and Its Application to Numerical Analysis", *Japan J. Indust. Appl. Math.* 26, 125143, 2009, (reprinted from Research Association of Applied Geometry (RAAG) Memoirs, Vol. 2, pp. 2946, 1958).

A new algebraic structure on intervals - Why?

The algebraic structure of intervals matters

T. Sunaga, “Theory of an Interval Algebra and Its Application to Numerical Analysis”, *Japan J. Indust. Appl. Math.* 26, 125143, 2009, (reprinted from Research Association of Applied Geometry (RAAG) Memoirs, Vol. 2, pp. 2946, 1958).

M. A. Sainz, J. Armengol, R. Calm, P. Herrero, L. Jorba, J. Vehi, “Modal Interval Analysis - New Tools for Numerical Information”, Lecture Notes in Mathematics, Springer International Publishing Switzerland 2014.

A new algebraic structure on intervals - Why?

The algebraic structure of intervals matters

T. Sunaga, "Theory of an Interval Algebra and Its Application to Numerical Analysis", *Japan J. Indust. Appl. Math.* 26, 125143, 2009, (reprinted from Research Association of Applied Geometry (RAAG) Memoirs, Vol. 2, pp. 2946, 1958).

M. A. Sainz, J. Armengol, R. Calm, P. Herrero, L. Jorba, J. Vehi, "Modal Interval Analysis - New Tools for Numerical Information", Lecture Notes in Mathematics, Springer International Publishing Switzerland 2014.

Endow the set of intervals with the median structure

Concepts, History and Application

Foundational concepts for this work include lattices, semilattices, distributive lattices and semilattices, abstract algebras, ternary operations, median algebraic structures ...

Donald Knuth reports that Emil Post while working on his Ph.D. Thesis (Columbia University, 1920) proved that: Every monotone, self-dual Boolean function $f(x_1, \dots x_n)$ can be expressed entirely in terms of the median operation denoted $\langle xyz \rangle$.

Important work has been carried out by mathematicians such as Birkhoff, Bandelt, Bàrthelemy, Monjardet, Sholander.etc.

Applications concern, mainly, discrete structures such as graphs, trees, lattice structures and more recently concepts such as the formal theory of majority consensus, etc.

Basic definitions

A ternary algebra is a set M equipped with a single ternary operation $M^3 \to M : (a, b, c) \mapsto (abc)$

Basic definitions

A ternary algebra is a set M equipped with a single ternary operation

$$M^3 \to M : (a, b, c) \mapsto (abc)$$

M is a median algebra if the ternary operation obeys the following axioms:

Basic definitions

A ternary algebra is a set M equipped with a single ternary operation
$$M^3 \to M : (a, b, c) \mapsto (abc)$$

M is a median algebra if the ternary operation obeys the following axioms:

- $(a, a, b) = a$ (majority law)
- $(a, b, c) = (a, c, b) = \cdots = (c, b, a)$ (commutative law)
- $(a, b, (c, b, d)) = ((a, b, c), b, d)$ (associative law)

Basic definitions

A ternary algebra is a set M equipped with a single ternary operation $M^3 \to M : (a, b, c) \mapsto (abc)$

M is a median algebra if the ternary operation obeys the following axioms:

- $(a, a, b) = a$ (majority law)
- $(a, b, c) = (a, c, b) = \cdots = (c, b, a)$ (commutative law)
- $(a, b, (c, b, d)) = ((a, b, c), b, d)$ (associative law)

Some authors, (Bandelt and J. Hedlíkovà) for M to be a median algebra require the ternary operation to satisfy the following identities:

- $(aab) = a$ (idempotency)
- $(abc) = (bac) = (acb)$ (symmetry)
- $(ab(cde)) = ((abc)d(abe))$ (distributivity)

Basic definitions

A ternary algebra is a set M equipped with a single ternary operation $M^3 \to M \,:\, (a,b,c) \mapsto (abc)$

M is a median algebra if the ternary operation obeys the following axioms:

- $(a,a,b) = a$ (majority law)
- $(a,b,c) = (a,c,b) = \cdots = (c,b,a)$ (commutative law)
- $(a,b,(c,b,d)) = ((a,b,c),b,d)$ (associative law)

Some authors, (Bandelt and J. Hedlíkovà) for M to be a median algebra require the ternary operation to satisfy the following identities:

- $(aab) = a$ (idempotency)
- $(abc) = (bac) = (acb)$ (symmetry)
- $(ab(cde)) = ((abc)d(abe))$ (distributivity)

One is able to demonstrate that the distributivity identity can be derived by the previous axioms

Basic definitions

A ternary algebra is a set M equipped with a single ternary operation
$M^3 \to M : (a, b, c) \mapsto (abc)$

M is a median algebra if the ternary operation obeys the following axioms:

- $(a, a, b) = a$ (majority law)
- $(a, b, c) = (a, c, b) = \cdots = (c, b, a)$ (commutative law)
- $(a, b, (c, b, d)) = ((a, b, c), b, d)$ (associative law)

Some authors, (Bandelt and J. Hedlíkovà) for M to be a median algebra require the ternary operation to satisfy the following identities:

- $(aab) = a$ (idempotency)
- $(abc) = (bac) = (acb)$ (symmetry)
- $(ab(cde)) = ((abc)d(abe))$ (distributivity)

One is able to demonstrate that the distributivity identity can be derived by the previous axioms

Basic definitions (cont.)

If M is a distributive lattice $(M, \vee, \wedge)$ one can define a ternary operation $m : M^3 \to M$ such that for $a, b, c \in M$

Basic definitions (cont.)

If M is a distributive lattice $(M, \vee, \wedge)$ one can define a ternary operation $m : M^3 \to M$ such that for $a, b, c \in M$

$$m(a,b,c) = (a \vee b) \wedge (a \vee c) \wedge (b \vee c),$$

where the symbols $\wedge$ and $\vee$ denote the *join* and *meet* operations of the lattice M.

Basic definitions (cont.)

If M is a distributive lattice $(M, \vee, \wedge)$ one can define a ternary operation $m : M^3 \to M$ such that for $a, b, c \in M$

$$m(a, b, c) = (a \vee b) \wedge (a \vee c) \wedge (b \vee c),$$

where the symbols $\wedge$ and $\vee$ denote the *join* and *meet* operations of the lattice M.

This operation is referred as the median ternary operation in M and satisfies the axioms above.

Note that, this ternary operation is self dual in the sense that

$$(a \vee b) \wedge (a \vee c) \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \vee (b \wedge c).$$

Examples

Every chain is a distributive lattice and so are the real numbers $\mathbb{R}$.
Therefore, $\mathbb{R}$ is a median algebra.

Examples

Every chain is a distributive lattice and so are the real numbers $\mathbb{R}$. Therefore, $\mathbb{R}$ is a median algebra.

- $(0.5, 1, 8) = 1$
- $(0.5, 1, 8) = (0.5, 8, 1) = \cdots = (8, 0.5, 1) = 1$
- $(0.5, 1, (2.5, 1, 8)) = ((0.5, 1, 2.5), 1, 8) = 1$

Examples

Every chain is a distributive lattice and so are the real numbers $\mathbb{R}$. Therefore, $\mathbb{R}$ is a median algebra.

- $(0.5, 1, 8) = 1$
- $(0.5, 1, 8) = (0.5, 8, 1) = \cdots = (8, 0.5, 1) = 1$
- $(0.5, 1, (2.5, 1, 8)) = ((0.5, 1, 2.5), 1, 8) = 1$

The algebraic structure of intervals

Contrary to the real numbers $\mathbb{R}$ which is equipped with a total order, the set of real intervals $\mathbb{IR}$ is partially ordered.
This is also valid for the extended set of real intervals $\mathbb{IR}^*$ including the modal.

The algebraic structure of intervals

Contrary to the real numbers $\mathbb{R}$ which is equipped with a total order, the set of real intervals $\mathbb{IR}$ is partially ordered.
This is also valid for the extended set of real intervals $\mathbb{IR}^*$ including the modal.

Moreover, recalling the definitions of the "min" "max" operators on $(\mathbb{IR}^*, \leqslant)$ as well as of the operators "join" and "meet" on $(\mathbb{IR}^*, \subseteq)$ (Sainz et al. Modal Interval Analysis - New Tools for Numerical Information) we state that the set of extended real intervals $\mathbb{IR}^*$ including the modal is a distributive lattice.

Therefore, $\mathbb{IR}^*$ is a median algebra.

The algebraic structure of intervals

Contrary to the real numbers $\mathbb{R}$ which is equipped with a total order, the set of real intervals $\mathbb{IR}$ is partially ordered.
This is also valid for the extended set of real intervals $\mathbb{IR}^*$ including the modal.

Moreover, recalling the definitions of the "min" "max" operators on $(\mathbb{IR}^*, \leqslant)$ as well as of the operators "join" and "meet" on $(\mathbb{IR}^*, \subseteq)$ (Sainz et al. Modal Interval Analysis - New Tools for Numerical Information) we state that the set of extended real intervals $\mathbb{IR}^*$ including the modal is a distributive lattice.

Therefore, $\mathbb{IR}^*$ is a median algebra.

Introducing the median structure

Hence, we may define the ternary operation for this median algebra as follows:

Introducing the median structure

Hence, we may define the ternary operation for this median algebra as follows:

Definition

For any three intervals $([x], [y], [z]) \mapsto ([x][y][z])$

$$\begin{aligned}([x][y][z]) &= \max\{\min\{[x],[y]\}, \min\{[x],[z]\}, \min\{[y],[z]\}\} \\ &= \min\{\max\{[x],[y]\}, \max\{[x],[z]\}, \max\{[y],[z]\}\}\end{aligned}$$

$$\begin{aligned}([x][y][z]) &= ([x] \wedge [y]) \vee ([x] \wedge [z]) \vee ([y] \wedge [z]) \\ &= ([x] \vee [y]) \wedge ([x] \vee [z]) \wedge ([y] \vee [z])\end{aligned}$$

Introducing the median structure

Hence, we may define the ternary operation for this median algebra as follows:

Definition

For any three intervals $([x],[y],[z]) \mapsto ([x][y][z])$

$$
\begin{aligned}
([x][y][z]) &= \max\{\min\{[x],[y]\}, \min\{[x],[z]\}, \min\{[y],[z]\}\} \\
&= \min\{\max\{[x],[y]\}, \max\{[x],[z]\}, \max\{[y],[z]\}\}
\end{aligned}
$$

$$
\begin{aligned}
([x][y][z]) &= ([x] \wedge [y]) \vee ([x] \wedge [z]) \vee ([y] \wedge [z]) \\
&= ([x] \vee [y]) \wedge ([x] \vee [z]) \wedge ([y] \vee [z])
\end{aligned}
$$

One may easily verify that the previous equalities provide the same result.

Introducing the median structure (cont.)

In order to facilitate calculations we formulate the following:

Introducing the median structure (cont.)

In order to facilitate calculations we formulate the following:

Calculation rule

For any three intervals $[x] = [\underline{x}, \overline{x}], [y] = [\underline{y}, \overline{y}], [z] = [\underline{z}, \overline{z}] \in \mathbb{IR}$ we define the median interval $([x][y][z])$ to be the result of the ternary operation $\mathbb{IR}^3 \to \mathbb{IR}$, $([x], [y], [z]) \mapsto ([x][y][z]) = [w] = [\underline{w}, \overline{w}]$ such that:

$$\underline{w} = (\underline{x}, \underline{y}, \underline{z}) \text{ and } \overline{w} = (\overline{x}, \overline{y}, \overline{z})$$

Introducing the median structure (cont.)

In order to facilitate calculations we formulate the following:

Calculation rule

For any three intervals $[x] = [\underline{x}, \overline{x}], [y] = [\underline{y}, \overline{y}], [z] = [\underline{z}, \overline{z}] \in \mathbb{IR}$ we define the median interval $([x][y][z])$ to be the result of the ternary operation $\mathbb{IR}^3 \to \mathbb{IR}$, $([x], [y], [z]) \mapsto ([x][y][z]) = [w] = [\underline{w}, \overline{w}]$ such that:

$$\underline{w} = (\underline{x}, \underline{y}, \underline{z}) \text{ and } \overline{w} = (\overline{x}, \overline{y}, \overline{z})$$

The interval $[w]$ defined by the previous rule is the set $\{\mathrm{w} \in \mathbb{R} | \mathrm{w} = (\mathrm{xyz}) \text{ for some } \mathrm{x} \in [x], \mathrm{y} \in [y] \text{ and } \mathrm{z} \in [z]\}$

Introducing the median structure (cont.)

In order to facilitate calculations we formulate the following:

Calculation rule

For any three intervals $[x] = [\underline{x}, \overline{x}], [y] = [\underline{y}, \overline{y}], [z] = [\underline{z}, \overline{z}] \in \mathbb{IR}$ we define the median interval $([x][y][z])$ to be the result of the ternary operation $\mathbb{IR}^3 \to \mathbb{IR}$, $([x],[y],[z]) \mapsto ([x][y][z]) = [w] = [\underline{w}, \overline{w}]$ such that:

$$\underline{w} = (\underline{x}, \underline{y}, \underline{z}) \text{ and } \overline{w} = (\overline{x}, \overline{y}, \overline{z})$$

The interval $[w]$ defined by the previous rule is the set $\{\mathrm{w} \in \mathbb{R} | \mathrm{w} = (\mathrm{xyz}) \text{ for some } \mathrm{x} \in [x], \mathrm{y} \in [y] \text{ and } \mathrm{z} \in [z]\}$

Moreover, one may show that the interval $[w]$ is uniquely defined and contains the median of any triplet $(\mathrm{x}, \mathrm{y}, \mathrm{z})$ of elements of the intervals $[x], [y], [z]$ and only them.

Introducing the median structure (cont.)

In order to facilitate calculations we formulate the following:

Calculation rule

For any three intervals $[x] = [\underline{x}, \overline{x}], [y] = [\underline{y}, \overline{y}], [z] = [\underline{z}, \overline{z}] \in \mathbb{IR}$ we define the median interval $([x][y][z])$ to be the result of the ternary operation $\mathbb{IR}^3 \to \mathbb{IR}$, $([x],[y],[z]) \mapsto ([x][y][z]) = [w] = [\underline{w}, \overline{w}]$ such that:

$$\underline{w} = (\underline{x}, \underline{y}, \underline{z}) \text{ and } \overline{w} = (\overline{x}, \overline{y}, \overline{z})$$

The interval $[w]$ defined by the previous rule is the set $\{\mathrm{w} \in \mathbb{R} | \mathrm{w} = (\mathrm{xyz}) \text{ for some } \mathrm{x} \in [x], \mathrm{y} \in [y] \text{ and } \mathrm{z} \in [z]\}$

Moreover, one may show that the interval $[w]$ is uniquely defined and contains the median of any triplet $(\mathrm{x}, \mathrm{y}, \mathrm{z})$ of elements of the intervals $[x], [y], [z]$ and only them.

However, $[w]$ may or may not coincide with any of the intervals $[x], [y], [z]$.

Introducing the median structure (cont.)

In order to facilitate calculations we formulate the following:

Calculation rule

For any three intervals $[x] = [\underline{x}, \overline{x}], [y] = [\underline{y}, \overline{y}], [z] = [\underline{z}, \overline{z}] \in \mathbb{IR}$ we define the median interval $([x][y][z])$ to be the result of the ternary operation $\mathbb{IR}^3 \to \mathbb{IR}$, $([x], [y], [z]) \mapsto ([x][y][z]) = [w] = [\underline{w}, \overline{w}]$ such that:

$\underline{w} = (\underline{x}, \underline{y}, \underline{z})$ and $\overline{w} = (\overline{x}, \overline{y}, \overline{z})$

The interval $[w]$ defined by the previous rule is the set $\{\mathrm{w} \in \mathbb{R} | \mathrm{w} = (\mathrm{xyz}) \text{ for some } \mathrm{x} \in [x], \mathrm{y} \in [y] \text{ and } \mathrm{z} \in [z]\}$

Moreover, one may show that the interval $[w]$ is uniquely defined and contains the median of any triplet $(\mathrm{x}, \mathrm{y}, \mathrm{z})$ of elements of the intervals $[x], [y], [z]$ and only them.

However, $[w]$ may or may not coincide with any of the intervals $[x], [y], [z]$. For all the above definitions it is easy to verify the median algebra axioms.

Introducing the median structure (cont.)

In order to facilitate calculations we formulate the following:

Calculation rule

For any three intervals $[x] = [\underline{x}, \overline{x}], [y] = [\underline{y}, \overline{y}], [z] = [\underline{z}, \overline{z}] \in \mathbb{IR}$ we define the median interval $([x][y][z])$ to be the result of the ternary operation $\mathbb{IR}^3 \to \mathbb{IR}$, $([x], [y], [z]) \mapsto ([x][y][z]) = [w] = [\underline{w}, \overline{w}]$ such that:

$$\underline{w} = (\underline{x}, \underline{y}, \underline{z}) \text{ and } \overline{w} = (\overline{x}, \overline{y}, \overline{z})$$

The interval $[w]$ defined by the previous rule is the set $\{\mathrm{w} \in \mathbb{R} | \mathrm{w} = (\mathrm{xyz}) \text{ for some } \mathrm{x} \in [x], \mathrm{y} \in [y] \text{ and } \mathrm{z} \in [z]\}$

Moreover, one may show that the interval $[w]$ is uniquely defined and contains the median of any triplet $(\mathrm{x}, \mathrm{y}, \mathrm{z})$ of elements of the intervals $[x], [y], [z]$ and only them.

However, $[w]$ may or may not coincide with any of the intervals $[x], [y], [z]$. For all the above definitions it is easy to verify the median algebra axioms.

Example of practical computations

Let us consider the intervals $[2, 1]$, $[-1, 0]$ and $[-3, 3]$. The median

$$([2, 1], [-1, 0], [-3, 3]) = [-1, 1]$$

can be computed with the following different ways:

Example of practical computations

Let us consider the intervals $[2, 1]$, $[-1, 0]$ and $[-3, 3]$. The median

$$([2, 1], [-1, 0], [-3, 3]) = [-1, 1]$$

can be computed with the following different ways:

- $[(2, -1, -3), (1, 0, 3)] = [-1, 1]$

Example of practical computations

Let us consider the intervals $[2,1]$, $[-1,0]$ and $[-3,3]$. The median

$$([2,1],[-1,0],[-3,3]) = [-1,1]$$

can be computed with the following different ways:

- $[(2,-1,-3),(1,0,3)] = [-1,1]$
- $\max\{\min\{[2,1],[-1,0]\},\min\{[2,1],[-3,3]\},\min\{[-1,0],[-3,3]\}\} =$
  $\max\{[\min\{2,-1\},\min\{1,0\}],[\min\{2,-3\},\min\{1,3\}],$
  $[\min\{-1,-3\},\min\{0,3\}]\} =$
  $\max\{[-1,0],[-3,1],[-3,0]\} =$
  $[\max\{-1,-3,-3\},\max\{0,1,0\}] =$
  $[-1,1]$

Example of practical computations

Let us consider the intervals $[2,1]$, $[-1,0]$ and $[-3,3]$. The median

$$([2,1],[-1,0],[-3,3]) = [-1,1]$$

can be computed with the following different ways:

- $[(2,-1,-3),(1,0,3)] = [-1,1]$
- $\max\{\min\{[2,1],[-1,0]\},\min\{[2,1],[-3,3]\},\min\{[-1,0],[-3,3]\}\} =$
  $\max\{[\min\{2,-1\},\min\{1,0\}],[\min\{2,-3\},\min\{1,3\}],$
  $[\min\{-1,-3\},\min\{0,3\}]\} =$
  $\max\{[-1,0],[-3,1],[-3,0]\} =$
  $[\max\{-1,-3,-3\},\max\{0,1,0\}] =$
  $[-1,1]$
- $\min\{\max\{[2,1],[-1,0]\},\max\{[2,1],[-3,3]\},\max\{[-1,0],[-3,3]\}\} =$
  $\min\{[\max\{2,-1\},\max\{1,0\}],[\max\{2,-3\},\max\{1,3\}],$
  $[\max\{-1,-3\},\max\{0,3\}]\} =$
  $\min\{[2,1],[2,3],[-1,3]\} =$
  $[\min\{2,2,-1\},\min\{1,3,3\}] =$
  $[-1,1]$

Example of practical computations (cont.)

Finally, using the “join” and ”meet” operations we get:

Example of practical computations (cont.)

Finally, using the “join” and ”meet” operations we get:

- $([2,1] \wedge [-1,0]) \vee ([2,1] \wedge [-3,3]) \vee ([-1,0] \wedge [-3,3]) =$
  $[\max\{2,-1\}, \min\{1,0\}] \vee [\max\{2,-3\}, \min\{1,3\}] \vee$
  $[\max\{-1,-3\}, \min\{0,3\}]\} =$
  $[2,0] \vee [2,1] \vee [-1,0] =$
  $[\min\{2,2,-1\}, \max\{0,1,0\}] =$
  $[-1,1]$

Example of practical computations (cont.)

Finally, using the “join” and ”meet” operations we get:

- $([2,1] \wedge [-1,0]) \vee ([2,1] \wedge [-3,3]) \vee ([-1,0] \wedge [-3,3]) =$
  $[\max\{2,-1\}, \min\{1,0\}] \vee [\max\{2,-3\}, \min\{1,3\}] \vee$
  $[\max\{-1,-3\}, \min\{0,3\}]\} =$
  $[2,0] \vee [2,1] \vee [-1,0] =$
  $[\min\{2,2,-1\}, \max\{0,1,0\}] =$
  $[-1,1]$
- $([2,1] \vee [-1,0]) \wedge ([2,1] \vee [-3,3]) \wedge ([-1,0] \vee [-3,3]) =$
  $[\min\{2,-1\}, \max\{1,0\}] \wedge [\min\{2,-3\}, \max\{1,3\}] \wedge$
  $[\min\{-1,-3\}, \max\{0,3\}]\} =$
  $[-1,1] \wedge [-3,3] \wedge [-3,3] =$
  $[\max\{-1,-3,-3\}, \min\{1,3,3\}] =$
  $[-1,1]$

Median of interval vectors

The previous considerations and results generalize for vectors of intervals, in a natural way.

In this case the ternary operation, of any form, on real intervals apply to each dimension of the vector thus, producing a vector of median intervals.

Hence, let us consider **x, y, z** $\in \mathbb{IR}^n$ that is 3 n-dimensional intervals.

Median of interval vectors

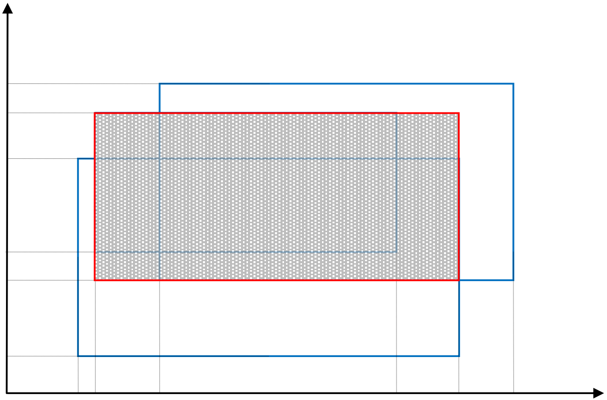
The previous considerations and results generalize for vectors of intervals, in a natural way.
In this case the ternary operation, of any form, on real intervals apply to each dimension of the vector thus, producing a vector of median intervals.
Hence, let us consider **x, y, z** $\in \mathbb{IR}^n$ that is 3 n-dimensional intervals.

$$\mathbf{x} = ([\mathrm{x}_1], [\mathrm{x}_2], \ldots, [\mathrm{x}_n]) = ([\underline{x}_1, \overline{x}_1], [\underline{x}_2, \overline{x}_2], \ldots, [\underline{x}_n, \overline{x}_n])$$
$$\mathbf{y} = ([\mathrm{y}_1], [\mathrm{y}_2], \ldots, [\mathrm{y}_n]) = ([\underline{y}_1, \overline{y}_1], [\underline{y}_2, \overline{y}_2], \ldots, [\underline{y}_n, \overline{y}_n])$$
$$\mathbf{z} = ([\mathrm{z}_1], [\mathrm{z}_2], \ldots, [\mathrm{z}_n]) = ([\underline{z}_1, \overline{z}_1], [\underline{z}_2, \overline{z}_2], \ldots, [\underline{z}_n, \overline{z}_n])$$

$$(\mathbf{x}, \mathbf{y}, \mathbf{z}) = (([\mathrm{x}_1], [\mathrm{y}_1], [\mathrm{z}_1]), ([\mathrm{x}_2], [\mathrm{y}_2], [\mathrm{z}_2]) \ldots ([\mathrm{x}_n], [\mathrm{y}_n], [\mathrm{z}_n])) =$$
$$([(\underline{x}_1, \underline{y}_1, \underline{z}_1), (\overline{x}_1, \overline{y}_1, \overline{z}_1)], [(\underline{x}_2, \underline{y}_2, \underline{z}_2), (\overline{x}_2, \overline{y}_2, \overline{z}_2)] \ldots$$
$$[(\underline{x}_n, \underline{y}_n, \underline{z}_n), (\overline{x}_n, \overline{y}_n, \overline{z}_n)])$$

Median of interval vectors

The interval median in 2 dimensions is illustrated by the “textured” rectangle.



Interval median and interval arithmetic vectors (proper intervals only)

Interval addition is distributive with respect to the interval median. It is straightforward to show that for any proper or point intervals $a, b, c, d \in \mathbb{IR}$ the following equalities hold true:

$$a + (b, c, d) = (a + b, a + c, a + d) = (b, c, d) + a$$

Interval median and interval arithmetic vectors (proper intervals only)

Interval addition is distributive with respect to the interval median. It is straightforward to show that for any proper or point intervals $a, b, c, d \in \mathbb{IR}$ the following equalities hold true:

$$a + (b, c, d) = (a + b, a + c, a + d) = (b, c, d) + a$$

Interval subtraction is also distributive with respect to the interval median. It is straightforward to show that for any proper or point intervals $a, b, c, d \in \mathbb{IR}$ the following equalities hold true:

$$a - (b, c, d) = (a - b, a - c, a - d)$$

and

$$(b, c, d) - a = (b - a, c - a, d - a)$$

Interval median and interval arithmetic vectors (proper intervals only)

Interval multiplication is not always distributive with respect to the interval median.

It is straightforward to show that $a \cdot (b, c, d) = (a \cdot b, a \cdot c, a \cdot d)$ if a is a point interval or when the bounds of a have the same sign, otherwise multiplication is sub-distributive i.e. $a \cdot (b, c, d) \subseteq (a \cdot b, a \cdot c, a \cdot d)$

Interval median and interval arithmetic vectors (proper intervals only)

Interval multiplication is not always distributive with respect to the interval median.

It is straightforward to show that $a \cdot (b, c, d) = (a \cdot b, a \cdot c, a \cdot d)$ if a is a point interval or when the bounds of a have the same sign, otherwise multiplication is sub-distributive i.e. $a \cdot (b, c, d) \subseteq (a \cdot b, a \cdot c, a \cdot d)$

Interval division by an interval not containing zero as an element is not always distributive with respect to interval median.

Division behaves the same way as multiplication.

Interval median and interval relation-containment operations

Interval median is compatible with relational and containment operations on intervals. Hence, one may easily show the following:

If $a_1, b_1, c_1, a_2, b_2, c_2 \in \mathbb{IR}^*$ such that $a_1 \subseteq a_2$ and $b_1 \subseteq b_2$ ans $c_1 \subseteq c_2$ then

$$(a_1, b_1, c_1) \subseteq (a_2, b_2, c_2).$$

Note that the same stands if instead of the operator "$\subseteq$" one considers "$\subset$" or even "$<$" and "$\leq$".

Computing the median of a sequence of interval data

Analysis of interval data has received increasing interest over the last two decades. This trend has mobilized and validated existing work on interval statistics.

Computing the median of a sequence of interval data

Analysis of interval data has received increasing interest over the last two decades. This trend has mobilized and validated existing work on interval statistics.

Research contributing in this direction:

Computing the median of a sequence of interval data

Analysis of interval data has received increasing interest over the last two decades. This trend has mobilized and validated existing work on interval statistics.

Research contributing in this direction:

- Symbolic Data Analysis and the SODAS software. Edwin Diday et al, Universitè de Paris IX - Dauphine

Computing the median of a sequence of interval data

Analysis of interval data has received increasing interest over the last two decades. This trend has mobilized and validated existing work on interval statistics.

Research contributing in this direction:

- Symbolic Data Analysis and the SODAS software. Edwin Diday et al, Universitè de Paris IX - Dauphine
- Principal Component Analysis for Interval-Valued Observations. A. Douzal-Chouakria, L. Billard and E. Diday

Computing the median of a sequence of interval data

Analysis of interval data has received increasing interest over the last two decades. This trend has mobilized and validated existing work on interval statistics.

Research contributing in this direction:

- Symbolic Data Analysis and the SODAS software. Edwin Diday et al, Universitè de Paris IX - Dauphine
- Principal Component Analysis for Interval-Valued Observations. A. Douzal-Chouakria, L. Billard and E. Diday
- A PCA for interval-valued data based on midpoints and radii. F. Palumbo and C. N. Lauro

Computing the median of a sequence of interval data

Analysis of interval data has received increasing interest over the last two decades. This trend has mobilized and validated existing work on interval statistics.

Research contributing in this direction:

- Symbolic Data Analysis and the SODAS software. Edwin Diday et al, Universitè de Paris IX - Dauphine
- Principal Component Analysis for Interval-Valued Observations. A. Douzal-Chouakria, L. Billard and E. Diday
- A PCA for interval-valued data based on midpoints and radii. F. Palumbo and C. N. Lauro
- Basic Statistical Methods for Interval Data. F. Gioia, C. N. Lauro

Computing the median of a sequence of interval data

Analysis of interval data has received increasing interest over the last two decades. This trend has mobilized and validated existing work on interval statistics.

Research contributing in this direction:

- Symbolic Data Analysis and the SODAS software. Edwin Diday et al, Universitè de Paris IX - Dauphine
- Principal Component Analysis for Interval-Valued Observations. A. Douzal-Chouakria, L. Billard and E. Diday
- A PCA for interval-valued data based on midpoints and radii. F. Palumbo and C. N. Lauro
- Basic Statistical Methods for Interval Data. F. Gioia, C. N. Lauro
- An Interval Nonparametric Regression Method. Roberta A. de A. Fagundes et al.

Computing the median of a sequence of interval data (cont.)

Theoretical support for the approach presented hereafter is provided by the work of Hans-Jürgen Bandelt and Gerasimos C. Meletiou, "The algebra of majority consensus."

Theorem 1

Let $x_1, \ldots, x_n \mapsto (x_1 \cdots x_n)$ be a symmetric n-ary operation $(n \geqslant 5)$ on a set M such that the following two identities hold for some integer s with $(\frac{1}{2}n < s \leqslant \frac{2}{3}n)$:

1. $(w_1 \cdots w_{n-s} x^s) = x,$
2. $(w_1 \cdots w_{n-1}(x_1 \cdots x_n)) = ((w_1 \cdots w_{n-1} x_1) \cdots (w_1 \cdots w_{n-1} x_s) x_{s+1} \cdots x_n)$

for all $w_i, x_i, x \in M$. Then M is a median algebra with respect to the ternary operation $(x, y, z) \mapsto (xyz)$ defined by $(xyz) = (x^{n-s} y^{n-s} z^{2s-n})$.

Computing the median of a sequence of interval data (cont.)

The second important theorem from the same paper is the following.

Theorem 2

Let M be an n-ary algebra satisfying the conditions of Theorem 1. Assume that $r > \frac{1}{2}n$ is the smallest integer such that

$(x^r y^{n-r}) = x$ for all $x, y \in M$.

Then the clones of M and the associated median algebra coincide if and only if $n = 2r - 1$. In this case the n-ary median is obtained from the $(n-2)$-ary and ternary medians via the recurrence

$$(x_1 \cdots x_{2r-1}) = (x_1 \cdots x_{r-2}(x_{r-1}x_r x_{r+1})(x_{r-1}x_r x_{r+1}x_{r+2}^2) \\ \cdots (x_{r-1} \cdots x_{2r-2}x_{2r-1}^{r-1})), \text{ with } x_i \in M,$$

where notation $(w_1 \cdots w_{i+1} x^i)$ stands for i applications of the ternary with x

$((\cdots((w_1 w_2 x) w_3 x) \cdots) w_{i+1} x)$, with $w_i, x \in M$.

Computing the median of a sequence of interval data (cont.)

From a computational complexity point of view
if the number of elements is an odd number $n = 2r - 1$
then the formula of Theorem 2 implies that

$$\binom{r+1}{3}$$

ternary median operations are needed in order to compute the n-ary median.

Example for $n = 5$

$(x_1x_2x_3x_4x_5) = (x_1(x_2x_3x_4)((x_2x_3x_5)x_4x_5))$.

If $n = 7$ then

$$(x_1x_2x_3x_4x_5x_6x_7) = ((((x_3x_4x_7)x_5x_7)x_6x_7)(x_1x_2(x_3x_4x_5))$$
$$((x_1x_2((x_3x_4x_6)x_5x_6))(x_3x_4x_5)((x_3x_4x_6)x_5x_6))).$$

Computing the median of a sequence of interval data (cont.)

Practical application examples with the intervals:

- $([-2, -1], [0, 1], [1, 2], [3, 4], [5, 6]) =$

Computing the median of a sequence of interval data (cont.)

Practical application examples with the intervals:

- $([-2,-1],[0,1],[1,2],[3,4],[5,6]) =$
  $([-2,-1],([0,1],[1,2],[3,4]),(([0,1],[1,2],[5,6]),[3,4],[5,6)) =$

Computing the median of a sequence of interval data (cont.)

Practical application examples with the intervals:

- $([-2,-1],[0,1],[1,2],[3,4],[5,6]) =$
  $([-2,-1],([0,1],[1,2],[3,4]),(([0,1],[1,2],[5,6]),[3,4],[5,6)) =$
  $([-2-1],[1,2],([1,2],[3,4],[5,6])) =$

Computing the median of a sequence of interval data (cont.)

Practical application examples with the intervals:

- $([-2,-1],[0,1],[1,2],[3,4],[5,6]) =$
  $([-2,-1],([0,1],[1,2],[3,4]),(([0,1],[1,2],[5,6]),[3,4],[5,6)) =$
  $([-2-1],[1,2],([1,2],[3,4],[5,6])) =$
  $([-2,-1],[1,2],[3,4]) =$
  $[1,2]$

Computing the median of a sequence of interval data (cont.)

Practical application examples with the intervals:

- $([-2,-1],[0,1],[1,2],[3,4],[5,6]) =$
  $([-2,-1],([0,1],[1,2],[3,4]),(([0,1],[1,2],[5,6]),[3,4],[5,6)) =$
  $([-2-1],[1,2],([1,2],[3,4],[5,6])) =$
  $([-2,-1],[1,2],[3,4]) =$
  $[1,2]$
- $([0,1],[-1,1],[2,0],[-2,-1],[1,4]) =$

Computing the median of a sequence of interval data (cont.)

Practical application examples with the intervals:

- $([-2,-1],[0,1],[1,2],[3,4],[5,6]) =$
  $([-2,-1],([0,1],[1,2],[3,4]),(([0,1],[1,2],[5,6]),[3,4],[5,6)) =$
  $([-2-1],[1,2],([1,2],[3,4],[5,6])) =$
  $([-2,-1],[1,2],[3,4]) =$
  $[1,2]$
- $([0,1],[-1,1],[2,0],[-2,-1],[1,4]) =$
  $([0,1],([-1,1],[2,0],[-2,-1]),(([-1,1],[2,0],[1,4]),[-2,-1],[1,4)) =$

Computing the median of a sequence of interval data (cont.)

Practical application examples with the intervals:

- $([-2,-1],[0,1],[1,2],[3,4],[5,6]) =$
  $([-2,-1],([0,1],[1,2],[3,4]),(([0,1],[1,2],[5,6]),[3,4],[5,6)) =$
  $([-2-1],[1,2],([1,2],[3,4],[5,6])) =$
  $([-2,-1],[1,2],[3,4]) =$
  $[1,2]$

- $([0,1],[-1,1],[2,0],[-2,-1],[1,4]) =$
  $([0,1],([-1,1],[2,0],[-2,-1]),(([-1,1],[2,0],[1,4]),[-2,-1],[1,4)) =$
  $([0,1],([(-1,2,-2),(1,0,-1)]),(([(-1,2,1),(1,0,4)]),[-2,-1],[1,4])) =$

Computing the median of a sequence of interval data (cont.)

Practical application examples with the intervals:

- $([-2,-1],[0,1],[1,2],[3,4],[5,6]) =$
  $([-2,-1],([0,1],[1,2],[3,4]),(([0,1],[1,2],[5,6]),[3,4],[5,6)) =$
  $([-2-1],[1,2],([1,2],[3,4],[5,6])) =$
  $([-2,-1],[1,2],[3,4]) =$
  $[1,2]$

- $([0,1],[-1,1],[2,0],[-2,-1],[1,4]) =$
  $([0,1],([-1,1],[2,0],[-2,-1]),(([-1,1],[2,0],[1,4]),[-2,-1],[1,4)) =$
  $([0,1],([(-1,2,-2),(1,0,-1)]),(([(-1,2,1),(1,0,4)]),[-2,-1],[1,4])) =$
  $([0,1],[-1,0],([1,1],[-2,-1],[1,4])) =$

Computing the median of a sequence of interval data (cont.)

Practical application examples with the intervals:

- $([-2,-1],[0,1],[1,2],[3,4],[5,6]) =$
  $([-2,-1],([0,1],[1,2],[3,4]),(([0,1],[1,2],[5,6]),[3,4],[5,6)) =$
  $([-2-1],[1,2],([1,2],[3,4],[5,6])) =$
  $([-2,-1],[1,2],[3,4]) =$
  $[1,2]$

- $([0,1],[-1,1],[2,0],[-2,-1],[1,4]) =$
  $([0,1],([-1,1],[2,0],[-2,-1]),(([-1,1],[2,0],[1,4]),[-2,-1],[1,4)) =$
  $([0,1],([(-1,2,-2),(1,0,-1)]),(([(-1,2,1),(1,0,4)]),[-2,-1],[1,4])) =$
  $([0,1],[-1,0],([1,1],[-2,-1],[1,4])) =$
  $([0,1],[-1,0],[(1,-2,1),(1,-1,4)]) =$

Computing the median of a sequence of interval data (cont.)

Practical application examples with the intervals:

- $([-2,-1],[0,1],[1,2],[3,4],[5,6]) =$
  $([-2,-1],([0,1],[1,2],[3,4]),(([0,1],[1,2],[5,6]),[3,4],[5,6)) =$
  $([-2-1],[1,2],([1,2],[3,4],[5,6])) =$
  $([-2,-1],[1,2],[3,4]) =$
  $[1,2]$

- $([0,1],[-1,1],[2,0],[-2,-1],[1,4]) =$
  $([0,1],([-1,1],[2,0],[-2,-1]),(([-1,1],[2,0],[1,4]),[-2,-1],[1,4)) =$
  $([0,1],([(-1,2,-2),(1,0,-1)]),(([(-1,2,1),(1,0,4)]),[-2,-1],[1,4])) =$
  $([0,1],[-1,0],([1,1],[-2,-1],[1,4])) =$
  $([0,1],[-1,0],[(1,-2,1),(1,-1,4)]) =$
  $([0,1],[-1,0],[1,1]) =$

Computing the median of a sequence of interval data (cont.)

Practical application examples with the intervals:

- $$([-2,-1],[0,1],[1,2],[3,4],[5,6]) =$$
  $$([-2,-1],([0,1],[1,2],[3,4]),(([0,1],[1,2],[5,6]),[3,4],[5,6)) =$$
  $$([-2-1],[1,2],([1,2],[3,4],[5,6])) =$$
  $$([-2,-1],[1,2],[3,4]) =$$
  $$[1,2]$$

- $$([0,1],[-1,1],[2,0],[-2,-1],[1,4]) =$$
  $$([0,1],([-1,1],[2,0],[-2,-1]),(([-1,1],[2,0],[1,4]),[-2,-1],[1,4)) =$$
  $$([0,1],([(-1,2,-2),(1,0,-1)]),(([(-1,2,1),(1,0,4)]),[-2,-1],[1,4])) =$$
  $$([0,1],[-1,0],([1,1],[-2,-1],[1,4])) =$$
  $$([0,1],[-1,0],[(1,-2,1),(1,-1,4)]) =$$
  $$([0,1],[-1,0],[1,1]) =$$
  $$[(0,-1,1),(1,0,1)] =$$

Computing the median of a sequence of interval data (cont.)

Practical application examples with the intervals:

- $([-2,-1],[0,1],[1,2],[3,4],[5,6]) =$
  $([-2,-1],([0,1],[1,2],[3,4]),(([0,1],[1,2],[5,6]),[3,4],[5,6)) =$
  $([-2-1],[1,2],([1,2],[3,4],[5,6])) =$
  $([-2,-1],[1,2],[3,4]) =$
  $[1,2]$

- $([0,1],[-1,1],[2,0],[-2,-1],[1,4]) =$
  $([0,1],([-1,1],[2,0],[-2,-1]),(([-1,1],[2,0],[1,4]),[-2,-1],[1,4)) =$
  $([0,1],([(-1,2,-2),(1,0,-1)]),(([(-1,2,1),(1,0,4)]),[-2,-1],[1,4])) =$
  $([0,1],[-1,0],([1,1],[-2,-1],[1,4])) =$
  $([0,1],[-1,0],[(1,-2,1),(1,-1,4)]) =$
  $([0,1],[-1,0],[1,1]) =$
  $[(0,-1,1),(1,0,1)] =$
  $[0,1]$

Computing the median of a sequence of interval data (cont.)

Practical application examples with the intervals:

- $([-2,-1],[0,1],[1,2],[3,4],[5,6]) =$
  $([-2,-1],([0,1],[1,2],[3,4]),(([0,1],[1,2],[5,6]),[3,4],[5,6)) =$
  $([-2-1],[1,2],([1,2],[3,4],[5,6])) =$
  $([-2,-1],[1,2],[3,4]) =$
  $[1,2]$

- $([0,1],[-1,1],[2,0],[-2,-1],[1,4]) =$
  $([0,1],([-1,1],[2,0],[-2,-1]),(([-1,1],[2,0],[1,4]),[-2,-1],[1,4)) =$
  $([0,1],([(-1,2,-2),(1,0,-1)]),(([(-1,2,1),(1,0,4)]),[-2,-1],[1,4]))$ =
  $([0,1],[-1,0],([1,1],[-2,-1],[1,4])) =$
  $([0,1],[-1,0],[(1,-2,1),(1,-1,4)]) =$
  $([0,1],[-1,0],[1,1]) =$
  $[(0,-1,1),(1,0,1)] =$
  $[0,1]$

Computing the median of a sequence of interval data (cont.)

Consider the case of even number of elements, i.e. $n = 2r$.
The result in this case is an interval of intervals.

Computing the median of a sequence of interval data (cont.)

Consider the case of even number of elements, i.e. $n = 2r$.
The result in this case is an interval of intervals.

It makes sense if one considers that this is alike the case of a sequence with an even number of reals.
Then the median is computed by taking the interval of the two middle elements. Example: $1, 2, 3, 4, 5, 6$.
The median is the midpoint of the interval $[3, 4] = 3.5$

Computing the median of a sequence of interval data (cont.)

Consider the case of even number of elements, i.e. $n = 2r$.
The result in this case is an interval of intervals.

It makes sense if one considers that this is alike the case of a sequence with an even number of reals.
Then the median is computed by taking the interval of the two middle elements. Example: $1, 2, 3, 4, 5, 6$.
The median is the midpoint of the interval $[3, 4] = 3.5$

In the case of a sequence of n intervals $x_1, x_2, \ldots, x_n$ with $n = 2r$ then, it is proven that in any lattice the following relation stands between the two expression of the "majority tern" :

$$\bigvee_{\substack{I \subseteq \{1,\ldots,n\} \\ \frac{n}{2} < |I| \leqslant \frac{n}{2}+1}} \bigwedge_{i \in I} x_i \quad < \quad \bigwedge_{\substack{I \subseteq \{1,\ldots,n\} \\ \frac{n}{2} < |I| \leqslant \frac{n}{2}+1}} \bigvee_{i \in I} x_i$$

Computing the median of a sequence of interval data (cont.)

Consider the case of even number of elements, i.e. $n = 2r$.
The result in this case is an interval of intervals.

It makes sense if one considers that this is alike the case of a sequence with an even number of reals.
Then the median is computed by taking the interval of the two middle elements. Example: $1, 2, 3, 4, 5, 6$.
The median is the midpoint of the interval $[3, 4] = 3.5$

In the case of a sequence of n intervals $x_1, x_2, \ldots, x_n$ with $n = 2r$ then, it is proven that in any lattice the following relation stands between the two expression of the "majority tern" :

$$\bigvee_{\substack{I \subseteq \{1,\ldots,n\} \\ \frac{n}{2} < |I| \leqslant \frac{n}{2}+1}} \bigwedge_{i \in I} x_i \quad < \quad \bigwedge_{\substack{I \subseteq \{1,\ldots,n\} \\ \frac{n}{2} < |I| \leqslant \frac{n}{2}+1}} \bigvee_{i \in I} x_i$$

This relation gives the bounds of an interval of intervals whose mean value provides the median of the sequence.

Open problems and questions

Computation of the median of a sequence of interval data raises the following questions:

- Is it possible to derive a recursive algorithm for computing the median of n intervals based on $n-1$ or $n-2$ previous intervals (depending on n)?

Open problems and questions

Computation of the median of a sequence of interval data raises the following questions:

- Is it possible to derive a recursive algorithm for computing the median of n intervals based on $n-1$ or $n-2$ previous intervals (depending on n)?
- Derivation of a method using a formula for computing the median of an even number of intervals. The formula should be similar to the formula provided by Theorem 2 above.

Open problems and questions

Computation of the median of a sequence of interval data raises the following questions:

- Is it possible to derive a recursive algorithm for computing the median of n intervals based on $n-1$ or $n-2$ previous intervals (depending on n)?
- Derivation of a method using a formula for computing the median of an even number of intervals. The formula should be similar to the formula provided by Theorem 2 above.
- Is it possible to derive a unique algorithm for computing both an odd and an even number of intervals?

Open problems and questions

Computation of the median of a sequence of interval data raises the following questions:

- Is it possible to derive a recursive algorithm for computing the median of n intervals based on $n-1$ or $n-2$ previous intervals (depending on n)?
- Derivation of a method using a formula for computing the median of an even number of intervals. The formula should be similar to the formula provided by Theorem 2 above.
- Is it possible to derive a unique algorithm for computing both an odd and an even number of intervals?
- Is it possible to derive formulae for computing any type of quantiles of interval sequences?

Other possible applications of the interval median algebra

- Interval statistics such as quantiles and quantile regression.

Other possible applications of the interval median algebra

- Interval statistics such as quantiles and quantile regression.
- Interval median metric spaces.

Other possible applications of the interval median algebra

- Interval statistics such as quantiles and quantile regression.
- Interval median metric spaces.
- Interval median clustering.

Other possible applications of the interval median algebra

- Interval statistics such as quantiles and quantile regression.
- Interval median metric spaces.
- Interval median clustering.
- ...

Review of the literature on interval statistics, median metric spaces, median clustering, etc.

End of Presentation

Thank you for your attention.

Questions and/or Comments?